

Asymptotics for the norm of Bethe eigenstates in the periodic totally asymmetric exclusion process

Sylvain Prolhac

March 17, 2015

Abstract The normalization of Bethe eigenstates for the totally asymmetric simple exclusion process on a ring of L sites is studied, in the large L limit with finite density of particles, for all the eigenstates responsible for the relaxation to the stationary state on the KPZ time scale $T \sim L^{3/2}$. In this regime, the normalization is found to be essentially equal to the exponential of the action of a scalar free field. The large L asymptotics is obtained using the Euler-Maclaurin formula for summations on segments, rectangles and triangles, with various singularities at the borders of the summation range.

Keywords TASEP · Bethe ansatz · Euler-Maclaurin

PACS 02.30.Ik · 02.50.Ga · 05.40.-a · 05.60.Cd

1 Introduction

Understanding the large scale evolution of macroscopic systems from their microscopic dynamics is one of the central aims of statistical physics out of equilibrium. Much progress has been happening toward this goal for systems in the one-dimensional KPZ universality class [1, 2, 3, 4], which describes the fluctuations in some specific regimes for the height of the interface in growth models, the current of particles in driven diffusive systems and the free energy for directed polymers in random media.

The totally asymmetric simple exclusion process (TASEP) [5, 6] belongs to KPZ universality. On the infinite line, the current fluctuations in the long time limit [7] are equal to the ones that have been obtained from other models, in particular polynuclear growth model [8], directed polymer in random media [9], and from the Kardar-Parisi-Zhang equation [10] itself using the replica method [11, 12]. On a finite system, the stationary large deviations of the current for periodic TASEP [13] agree with the ones from the replica method [14] and with the ones for open TASEP at the transition separating the maximal current phase with the high and low density phases [15].

Much less is currently known about the crossover between fluctuations on the infinite line and in a finite system, see however [16, 17, 18, 19, 20, 21, 22]. The crossover takes place on the relaxation scale with times T of order $L^{3/2}$ characteristic of KPZ universality in $1+1$ dimension. The aim of the present paper is to compute the large L limit of the normalization of the Bethe eigenstates of TASEP that contribute to the relaxation regime. Our main result is that this limit depends on the eigenstate essentially through the free action of a field φ built by summing over elementary excitations corresponding to the eigenstate. This result can be used to derive an exact formula for the current fluctuations in the relaxation regime [23].

S. Prolhac

Laboratoire de Physique Théorique, IRSAMC, UPS, Université de Toulouse, France

Laboratoire de Physique Théorique, UMR 5152, Toulouse, CNRS, France

E-mail: sylvain.prolhac@irsamc.ups-tlse.fr

The paper is organized as follows. In section 2, we briefly recall the master equation generating the time evolution of TASEP and its deformation which allows to count the current of particles. In section 3, we summarize some known facts about Bethe ansatz for periodic TASEP, and state our main result (36) about the asymptotics of the norm of Bethe eigenstates. In section 4, we state the Euler-Maclaurin formula for summation on segments, triangles and rectangles with various singularities at the borders of the summation range. The Euler-Maclaurin formula is then used in section 5 to compute the asymptotics of the normalization of Bethe states. In appendix A, some properties of simple and double Hurwitz zeta functions are summarized.

2 Periodic TASEP

We consider TASEP with N hard-core particles on a periodic lattice of L sites. The continuous time dynamics consists of particle hopping from any site i to the next site $i + 1$ with rate 1 if the destination site is empty.

Since TASEP is a Markov process, the time evolution of the probability $P_T(\mathcal{C})$ to observe the system at time T in the configuration \mathcal{C} is generated by a master equation. A deformation of the master equation can be considered [13] to count the total integrated current of particles Q_T , defined as the total number of hops of particles up to time T . Defining $F_T(\mathcal{C}) = \sum_{Q=0}^{\infty} e^{\gamma Q} P_T(\mathcal{C}, Q)$ where $P_T(\mathcal{C}, Q)$ is the joint probability to have the system in configuration \mathcal{C} with $Q_T = Q$, one has

$$\frac{\partial}{\partial T} F_T(\mathcal{C}) = \sum_{\mathcal{C}' \neq \mathcal{C}} \left[e^{\gamma} w(\mathcal{C} \leftarrow \mathcal{C}') F_T(\mathcal{C}') - w(\mathcal{C}' \leftarrow \mathcal{C}) F_T(\mathcal{C}) \right]. \quad (1)$$

The hopping rate $w(\mathcal{C}' \leftarrow \mathcal{C})$ is equal to 1 if the configuration \mathcal{C}' can be obtained from \mathcal{C} by moving one particle from a site i to $i + 1$, and is equal to 0 otherwise. The deformed master equation (1) reduces to the usual master equation for the probabilities $P_T(\mathcal{C})$ when the fugacity γ is equal to 0. It can be encoded in a deformed Markov operator $M(\gamma)$ acting on the configuration space of dimension $\Omega = \binom{L}{N}$ in the sector with N particles. Gathering the $F_T(\mathcal{C})$ in a vector $|F_T\rangle$, one can write

$$\partial_T |F_T\rangle = M(\gamma) F_T. \quad (2)$$

The deformed master equation (1), (2) is known [13] to be integrable in the sense of quantum integrability, also called stochastic integrability [24] in the context of an evolution generated by a non-Hermitian stochastic operator. At $\gamma = 0$, the eigenvalue of the first excited state (gap) has been shown to scale as $L^{-3/2}$ using Bethe ansatz [25, 26, 27]. The whole spectrum has also been studied [28], and in particular the region with eigenvalues scaling as $L^{-3/2}$ [29]. In this article, we study the normalization of the corresponding eigenstates, which is needed for the calculation of fluctuations of Q_T on the relaxation scale $T \sim L^{3/2}$ [23]. There

$$\langle e^{\gamma Q_T} \rangle = \frac{\langle \mathcal{C} | e^{TM(\gamma)} | \mathcal{C}_0 \rangle}{\langle \mathcal{C} | e^{TM} | \mathcal{C}_0 \rangle} \quad (3)$$

is evaluated by inserting a decomposition of the identity operator in terms of left and right normalized eigenvectors. Throughout the paper, we consider the thermodynamic limit $L, N \rightarrow \infty$ with fixed density of particles

$$\rho = \frac{N}{L}, \quad (4)$$

and fixed rescaled fugacity

$$s = \sqrt{\rho(1 - \rho)} \gamma L^{3/2} \quad (5)$$

according to the relaxation scale $T \sim L^{3/2}$ in one-dimensional KPZ universality.

3 Bethe ansatz

In this section, we recall some known facts about Bethe ansatz for periodic TASEP, and state our main result about the asymptotics of the normalization of Bethe eigenstates.

3.1 Eigenvalues and eigenvectors

Bethe ansatz is one of the main tools that have been used to obtain exact results about dynamical properties of TASEP. It allows to diagonalize the N particle sector of the generator of the evolution $M(\gamma)$ in terms of N (complex) momenta q_j , $j = 1, \dots, N$. The eigenvectors are then written as sums over all $N!$ permutations assigning momenta to the particles. On the infinite line, the momenta are integrated over on some continuous curve in the complex plane [30]. For a finite system on the other hand, only a discrete set of N -tuples of momenta are allowed, as is usual for particles in a box. Writing $y_j = 1 - e^{\gamma} e^{iq_j}$, one can show that for the system with periodic boundary conditions, the complex numbers y_j , $j = 1, \dots, N$ have to satisfy the *Bethe equations*

$$e^{L\gamma}(1 - y_j)^L = (-1)^{N-1} \prod_{k=1}^N \frac{y_j}{y_k}. \quad (6)$$

We use the shorthand r to refer to the sets of N *Bethe roots* y_j solving the Bethe equations. The eigenstates are indexed by r . The corresponding eigenvalue of $M(\gamma)$ is equal to

$$E_r(\gamma) = \sum_{j=1}^N \frac{y_j}{1 - y_j}. \quad (7)$$

By translation invariance of the model, each eigenstate of $M(\gamma)$ is also eigenstate of the translation operator. The corresponding eigenvalue is

$$e^{2i\pi p_r/L} = e^{N\gamma} \prod_{j=1}^N (1 - y_j), \quad (8)$$

with total momentum $p_r \in \mathbb{Z}$.

The coefficients of the right and left (unnormalized) eigenvectors for a configuration with particles at positions $1 \leq x_1 < \dots < x_N \leq L$ are given by the determinants

$$\langle x_1, \dots, x_N | \psi_r(\gamma) \rangle = \det \left(y_k^{-j} (1 - y_k)^{x_j} e^{\gamma x_j} \right)_{j,k=1,\dots,N} \quad (9)$$

$$\langle \psi_r(\gamma) | x_1, \dots, x_N \rangle = \det \left(y_k^j (1 - y_k)^{-x_j} e^{-\gamma x_j} \right)_{j,k=1,\dots,N}. \quad (10)$$

These determinants are antisymmetric under the exchange of the y_j 's, and are thus divisible by the Vandermonde determinant of the y_j 's. In particular, for the configuration \mathcal{C}_X with particles at positions $(X, X+1, \dots, X+N-1)$, they reduce to

$$\langle \mathcal{C}_X | \psi_r(\gamma) \rangle = e^{\frac{2i\pi p_r X}{L}} e^{\frac{N(N-1)\gamma}{2}} \left(\prod_{j=1}^N y_j^{-N} \right) \prod_{1 \leq j < k \leq N} (y_j - y_k) \quad (11)$$

$$\langle \psi_r(\gamma) | \mathcal{C}_X \rangle = e^{-\frac{2i\pi p_r X}{L}} e^{-\frac{N(N-1)\gamma}{2}} \left(\prod_{j=1}^N \frac{y_j}{(1 - y_j)^{N-1}} \right) \prod_{1 \leq j < k \leq N} (y_k - y_j). \quad (12)$$

Based on numerical solutions, the expressions above for the eigenvectors and eigenvalues are only valid for generic values of γ . For specific values of γ , some eigenstates might be missing. Those can be identified, by adding a small perturbation to γ , as cases where several y_j 's coincide, which imply that the determinants in (9) and (10) vanish. This is in particular the case for the stationary eigenstate at $\gamma = 0$: in the limit $\gamma \rightarrow 0$, all y_j 's converge to 0 as $\gamma^{1/N}$. This will not be a problem here, as one can always add a small perturbation to γ when needed, see also [31] for a discussion in XXX and XXZ spin chain.

3.2 Normalization of Bethe eigenstates

The eigenvectors (9), (10) are not normalized. In order to write the decomposition of the identity

$$\mathbf{1} = \sum_r \frac{|\psi_r(\gamma)\rangle \langle \psi_r(\gamma)|}{\langle \psi_r(\gamma) | \psi_r(\gamma) \rangle}, \quad (13)$$

one needs to compute the scalar products $\langle \psi_r(\gamma) | \psi_r(\gamma) \rangle$ between left and right eigenstates corresponding to the same Bethe roots (and hence same eigenvalue).

Several results are known, both for on-shell (Bethe roots satisfying the Bethe equations) and off-shell (arbitrary y_j 's) scalar products. We write explicitly the dependency of the Bethe vectors (9) and (10) on the y_j 's as $\psi(\mathbf{y})$. For arbitrary complex numbers $y_j, w_j, j = 1, \dots, N$, it was shown [19,32] that

$$\langle \psi(\mathbf{w}) | \psi(\mathbf{y}) \rangle = \left(\prod_{j=1}^N \frac{1-y_j}{y_j} \frac{w_j^N}{(1-w_j)^L} \right) \det \left(\frac{\frac{(1-w_k)^L}{w_k^{N-1}} - \frac{(1-y_j)^L}{y_j^{N-1}}}{y_j - w_k} \right)_{j,k=1,\dots,N}. \quad (14)$$

For the mixed on-shell / off-shell case, where the y_j 's verify the Bethe equations while the w_j 's are arbitrary, one has the Slavnov determinant [33]

$$\langle \psi(\mathbf{w}) | \psi(\mathbf{y}) \rangle = (-1)^N \left(\prod_{j=1}^N \frac{(1-y_j)^{L+1}}{y_j^N (1-w_j)^L} \right) \left(\prod_{j=1}^N \prod_{k=1}^N (y_j - w_k) \right) \det \left(\partial_{y_i} \mathcal{E}(w_j, \mathbf{y}) \right)_{i,j=1,\dots,N}, \quad (15)$$

where the derivative with respect to y_i is taken before setting the y_j 's equal to solutions of the Bethe equations. The quantity $\mathcal{E}(\lambda, \mathbf{y})$ is the eigenvalue of the transfer matrix associated to TASEP with spectral parameter λ

$$\mathcal{E}(\lambda, \mathbf{y}) = \prod_{j=1}^N \frac{1}{1 - \lambda^{-1} y_j} + e^{L\gamma} (1 - \lambda)^L \prod_{j=1}^N \frac{1}{1 - \lambda y_j^{-1}}. \quad (16)$$

Finally, for left and right eigenvectors with the same on-shell Bethe roots y_j satisfying Bethe equations, the scalar product is equal to the Gaudin determinant [34,35]

$$\langle \psi_r(\gamma) | \psi_r(\gamma) \rangle = (-1)^N \left(\prod_{j=1}^N (1-y_j) \right) \det \left(\partial_{y_i} \log \left((1-y_j)^L \prod_{k=1}^N \frac{y_k}{y_j} \right) \right)_{i,j=1,\dots,N}. \quad (17)$$

Very similar determinantal expressions to (15) and (17) also exist for more general integrable models, in particular ASEP with particles hopping in both directions. The determinantal formula (14) for the fully off-shell case seems so far only available for the special case of TASEP.

In this paper, we only consider the on-shell scalar product (17), which can be simplified further by computing the derivative with respect to y_i and using the identity

$$\det(\alpha_i + \beta_i \delta_{i,j})_{i,j=1,\dots,N} = \left(\prod_{j=1}^N \beta_j \right) \left(1 + \sum_{j=1}^N \frac{\alpha_j}{\beta_j} \right), \quad (18)$$

where δ is Kronecker's delta symbol. The scalar product is then equal to

$$\langle \psi_r(\gamma) | \psi_r(\gamma) \rangle = \frac{L}{N} \left(\sum_{j=1}^N \frac{y_j}{N + (L-N)y_j} \right) \prod_{j=1}^N \left(L - N + \frac{N}{y_j} \right). \quad (19)$$

This normalization is somewhat arbitrary since it depends on the choice of the normalization in the definitions (9), (10). We consider then the configuration \mathcal{C}_X with particles at positions $(X, X+1, \dots, X+N-1)$ and define

$$\mathcal{N}_r(\gamma) = \Omega \frac{\langle \mathcal{C}_X | \psi_r(\gamma) \rangle \langle \psi_r(\gamma) | \mathcal{C}_X \rangle}{\langle \psi_r(\gamma) | \psi_r(\gamma) \rangle}, \quad (20)$$

with $\Omega = \binom{L}{N}$ the total number of configurations. One has

$$\mathcal{N}_r(\gamma) = (-1)^{\frac{N(N-1)}{2}} e^{-2i\pi p_r(\rho - \frac{1}{L})} \frac{e^{N(N-1)\gamma\Omega}}{N^N (\prod_{j=1}^N y_j)^{N-2}} \frac{\prod_{j=1}^N \prod_{k=j+1}^N (y_j - y_k)^2}{\left(\frac{1}{N} \sum_{j=1}^N \frac{y_j}{\rho + (1-\rho)y_j}\right) \prod_{j=1}^N \left(1 + \frac{1-\rho}{\rho} y_j\right)}. \quad (21)$$

Since this formula is based on the rather involved proof of (17) obtained in [35] for the slightly different case of the XXZ spin chain, we checked it numerically starting from (9), (10) for all systems with $2 \leq L \leq 10$, $1 \leq N \leq L-1$ and all eigenstates. We used the method described in the next section to solve the Bethe equations. Generic values were chosen for the parameter γ . Perfect agreement was found with (21).

In the basis of configurations, all the elements of the left stationary eigenvector at $\gamma = 0$ are equal since $M(0)$ is a stochastic matrix. The same is true for the right stationary eigenvector due to a property of pairwise balance verified by periodic TASEP [36]. Denoting the stationary state by the index 0, this implies $\mathcal{N}_0(0) = 1$.

The main goal of this article is the calculation of the asymptotics (36) of (21) for large L with fixed density of particles ρ and rescaled fugacity s for the first eigenstates beyond the stationary state.

3.3 Solution of the Bethe equations

The Bethe equations of TASEP can be solved in a rather simple way using the fact that they almost decouple, since the right hand side of (6) can be written as y_j^N times a symmetric function of the y_k 's independent of j . The strategy [25, 13] is then to give a name to that function of the y_k 's and treat it as a parameter independent of the Bethe roots, than is subsequently fixed using its explicit expression in terms of the y_k 's. This procedure can be conveniently written [37, 29] by introducing the function

$$g : y \mapsto \frac{1-y}{y^\rho}. \quad (22)$$

Indeed, defining the quantity

$$b = \gamma + \frac{1}{L} \sum_{j=1}^N \log y_j \quad (23)$$

and taking the power $1/L$ of the Bethe equations (6), we observe that there must exist wave numbers k_j , integers (half-integers) if N is odd (even) such that

$$g(y_j) = \exp\left(\frac{2i\pi k_j}{L} - b\right). \quad (24)$$

Inverting the function g leads to a rather explicit solution of the Bethe equations as

$$y_j = g^{-1}\left(\exp\left(\frac{2i\pi k_j}{L} - b\right)\right). \quad (25)$$

This expression is very convenient for large L asymptotic analysis using the Euler-Maclaurin formula.

3.4 First excited states

The stationary state corresponds to the choice $k_j = k_j^0$, $j = 1, \dots, N$ with

$$k_j^0 = j - \frac{N+1}{2}. \quad (26)$$

This choice closely resembles the Fermi sea of a system of spinless fermions.

We call *first excited states* the (infinitely many) eigenstates of $M(\gamma)$ having a real part scaling as $L^{-3/2}$ in the thermodynamic limit $L \rightarrow \infty$ with fixed density of particles ρ and purely imaginary rescaled

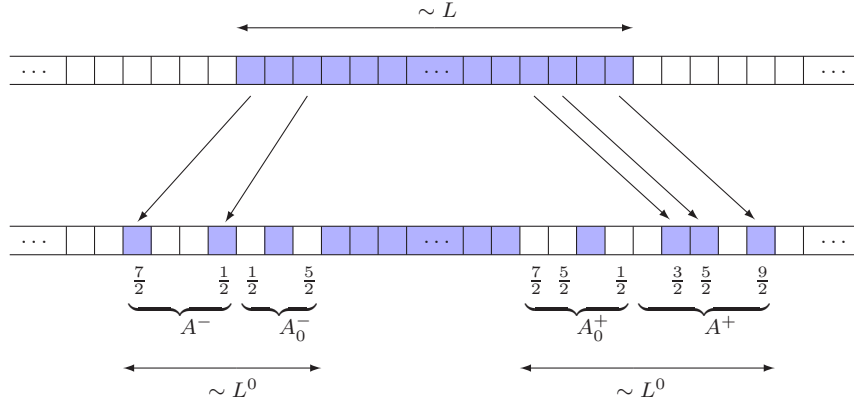


Fig. 1 Representation of the choices of the numbers k_j , $j = 1, \dots, N$ characterizing the first eigenstates. The red squares represent the k_j 's chosen. The upper line corresponds to the choice for the stationary state (26). The lower line corresponds to a generic eigenstate close to the stationary state, with excitations characterized by sets $A_0^- = \{\frac{1}{2}, \frac{5}{2}\}$, $A^- = \{\frac{1}{2}, \frac{7}{2}\}$, $A_0^+ = \{\frac{1}{2}, \frac{5}{2}, \frac{7}{2}\}$, $A^+ = \{\frac{3}{2}, \frac{5}{2}, \frac{9}{2}\}$ of cardinals $m_r^- = |A_0^-| = |A^-| = 2$ and $m_r^+ = |A_0^+| = |A^+| = 3$.

fugacity s . These eigenstates correspond to sets $\{k_j, j = 1, \dots, N\}$ close to the stationary choice (26). They are built by removing from $\{k_j^0, j = 1, \dots, N\}$ a finite number of k_j 's located at a finite distance of $\pm N/2$ and adding the same number of k_j 's at a finite distance of $\pm N/2$ outside of the interval $[-N/2, N/2]$. The first eigenstates are characterized by an equal number of k_j 's removed and added on each side. In particular, the choice $k_j = j - (N - 1)/2$ leads to a larger eigenvalue, $\text{Re } E(0) \sim L^{-2/3}$ [28], and thus does not belong to the first eigenstates. Numerical checks seem to support the fact that no other choices for the k_j 's lead to eigenvalues with a real part scaling as $L^{-3/2}$, although a proof of this is missing.

The first excited states can be described by four finite sets of positive half-integers $A_0^\pm, A^\pm \subset \mathbb{N} + \frac{1}{2}$: the set of k_j 's removed from (26) are $\{N/2 - a, a \in A_0^+\}$ and $\{-N/2 + a, a \in A_0^-\}$, while the set of k_j 's added are $\{N/2 + a, a \in A^+\}$ and $\{-N/2 - a, a \in A^-\}$, see figure 1. The cardinals of the sets verify the constraints

$$m_r^+ = |A_0^+| = |A^+| \quad \text{and} \quad m_r^- = |A_0^-| = |A^-|. \quad (27)$$

We call $m_r = m_r^+ + m_r^-$.

In the following, we use the notation r as a shorthand for (A_0^+, A^+, A_0^-, A^-) to refer to the corresponding excited state. The total momentum of an eigenstate, $p_r = \sum_{j=1}^N k_j$, can be written in terms of the four sets as $p_r = \sum_{a \in A_0^+} a + \sum_{a \in A^+} a - \sum_{a \in A_0^-} a - \sum_{a \in A^-} a$.

Only the excited states having an eigenvalue with real part scaling as $L^{-3/2}$ contribute to the relaxation for times $T \sim L^{3/2}$; the other eigenstates with larger eigenvalue only give exponentially small corrections when $L \rightarrow \infty$. This statement needs however some more justification since, in principle, it could be that the number of higher excited states becomes so large that $TE_r(\gamma)$ becomes negligible compared to the "entropy" of the spectrum in the expansion of (3) over the eigenstates. This entropy was studied in [28] at $\gamma = 0$ for the bulk of the spectrum with eigenvalues scaling proportionally to L . It was shown that the number of eigenvalues with a real part $-Le$ grows as $\exp(Ls(e))$ with $s(e) \sim e^{2/5}$ for small e . Assuming that the $2/5$ exponent still holds for eigenvalues scaling as $L^{-\alpha}$ with $-1 < \alpha < 3/2$, the contribution to (3) of the entropic part is of order $\exp(L^{(3-2\alpha)/5})$, which is always negligible compared to the contribution of $e^{TE_r(\gamma)} \sim \exp(L^{3/2-\alpha})$ except at $\alpha = 3/2$.

3.5 Field φ_r

The eigenvalue corresponding to the eigenstate r can be nicely written in terms of a function η_r [29]. From the result stated in section 3.8 about the normalization of Bethe states, the function $\varphi_r(u) =$

$-(2\pi)^{-3/2}\eta'_r(\frac{u}{2\pi})$ seems in fact the "good" object to describe the first excited states. It is defined by

$$\begin{aligned} \varphi_r(u) = & 2\sqrt{\pi} \left(e^{i\pi/4} \zeta\left(-\frac{1}{2}, \frac{1}{2} + \frac{iu}{2\pi}\right) + e^{-i\pi/4} \zeta\left(-\frac{1}{2}, \frac{1}{2} - \frac{iu}{2\pi}\right) \right) \\ & + i\sqrt{2} \left(\sum_{a \in A_0^+} \sqrt{u + 2i\pi a} + \sum_{a \in A^-} \sqrt{u + 2i\pi a} - \sum_{a \in A_0^-} \sqrt{u - 2i\pi a} - \sum_{a \in A^+} \sqrt{u - 2i\pi a} \right). \end{aligned} \quad (28)$$

The Hurwitz zeta function (137) can be seen as a kind of renormalization of an infinite contribution of the Fermi sea to φ_r . Indeed, using (47), (144) and introducing the quantities $\chi_a^\pm(u) = \pm i\sqrt{2u \pm 4i\pi a}$, we observe that φ_r can be written as a sum over momenta of elementary excitations k_j near $\pm N/2$ as

$$\varphi_r(u) = \lim_{M \rightarrow \infty} \left(-\frac{4\sqrt{2\pi}}{3} M^{3/2} + \frac{\sqrt{2}u}{\sqrt{\pi}} \sqrt{M} + \sum_{a \in B_M^+} \chi_a^-(u) + \sum_{a \in B_M^-} \chi_a^+(u) \right), \quad (29)$$

with $B_M^\pm = (\{-M + \frac{1}{2}, -M + \frac{3}{2}, \dots, -\frac{1}{2}\} \setminus (-A_0^\pm)) \cup A^\pm$.

The ζ functions are responsible for branch points $\pm i\pi$ for the function φ_r . The square roots provide additional branch points in $\pm 2i\pi(\mathbb{N} + \frac{1}{2})$. We define in the following φ_r with branch cuts $[i\pi, i\infty)$ and $(-\infty, -i\pi]$.

Using the relation between polylogarithms and Hurwitz zeta function, the field for the stationary state can be written as

$$\varphi_0(u) = -\frac{1}{\sqrt{2\pi}} \text{Li}_{3/2}(-e^u), \quad (30)$$

which is valid for $\text{Re } u < 0$, and for $\text{Re } u > 0$ with $|\text{Im } u| < \pi$.

Our main result (36), which expresses the asymptotics of the norm of Bethe eigenstates in terms of the free action of φ_r , seems to indicate that φ_r should be interpreted as a field, whose physical meaning is unclear at the moment.

3.6 Large L expansion of the parameter b

For all first excited states, the quantity b converges in the thermodynamic limit to b_0 [29], equal to

$$b_0 = \rho \log \rho + (1 - \rho) \log(1 - \rho). \quad (31)$$

Writing the correction as

$$b = b_0 + \frac{2\pi c}{L}, \quad (32)$$

a small generalization of [29] to non-zero rescaled fugacity s leads to the large L expansion

$$\varphi_r(2\pi c) \simeq s - \frac{2i\pi(1-2\rho)p_r}{3\sqrt{\rho(1-\rho)}\sqrt{L}}. \quad (33)$$

This expansion follows from applying the Euler-Maclaurin formula to (23), with Bethe roots y_j given by (25).

3.7 Large L expansion of the eigenvalues

Another small extension of [29] to nonzero rescaled fugacity s gives the expansion up to order $L^{-3/2}$ of the eigenvalue $E_r(\gamma)$ as

$$E_r(\gamma) \simeq \frac{s\sqrt{\rho(1-\rho)}}{\sqrt{L}} - \frac{2i\pi(1-2\rho)p_r}{L} + \frac{\sqrt{\rho(1-\rho)}}{L^{3/2}} \lim_{\Lambda \rightarrow \infty} \left(D^\Lambda + \int_{-\Lambda}^{2\pi c} du \varphi_r(u) \right), \quad (34)$$

where

$$D^\Lambda = \frac{4\sqrt{2}m_r}{3} \Lambda^{3/2} - 2\sqrt{2}i\pi \left(\sum_{a \in A_0^+} a + \sum_{a \in A^-} a - \sum_{a \in A_0^-} a - \sum_{a \in A^+} a \right) \sqrt{\Lambda} \quad (35)$$

cancels the divergent contribution from the integral. This expansion follows from the application of the Euler-Maclaurin formula to (7).

3.8 Large L expansion of the norm of Bethe eigenstates

In section 5, we derive the large L asymptotics (136) for the normalization of Bethe states, with b written as (32) and c arbitrary. Writing the solution of (33) at leading order in L as $2\pi c = \varphi_r^{-1}(s)$, the asymptotics of the normalization of Bethe states is obtained as

$$\begin{aligned} \mathcal{N}_r(\gamma) &\simeq e^{-2i\pi\rho p_r} e^{-s\sqrt{\rho(1-\rho)}\sqrt{L}} \\ &\times \frac{(\pi^2/4)^{m_r^2}}{(-4\pi^2)^{m_r}} \omega(A_0^+)^2 \omega(A_0^-)^2 \omega(A^+)^2 \omega(A^-)^2 \omega(A_0^+, A_0^-)^2 \omega(A^+, A^-)^2 \\ &\times \frac{e^{\varphi_r^{-1}(s)}}{\sqrt{2\pi} \varphi_r'(\varphi_r^{-1}(s))} \lim_{\Lambda \rightarrow \infty} \exp \left(-2m_r^2 \log \Lambda + \int_{-\Lambda}^{\varphi_r^{-1}(s)} du (\varphi_r'(u))^2 \right), \end{aligned} \quad (36)$$

with the combinatorial factors

$$\omega(A) = \prod_{\substack{a, a' \in A \\ a < a'}} (a - a') \quad \text{and} \quad \omega(A, A') = \prod_{a \in A} \prod_{a' \in A'} (a + a'). \quad (37)$$

This is the main technical result of the paper. The field φ_r is defined by (28).

One recovers the stationary value $\mathcal{N}_0(0) = 1$ using the fact that the solution c of $\varphi_0(2\pi c) = s$ for the stationary state goes to $-\infty$ when s goes to 0 and the expression (30) of φ_0 as a polylogarithm. For technical reasons, our derivation of (36) requires $\text{Re } c > 0$ for all the other eigenstates. This implies that $\text{Re } s$ can not be too small. Numerical resolution of $\varphi_r(2\pi c) = s$ for the first eigenstates seem to indicate that the condition $\text{Re } s \geq 0$ is always sufficient.

3.9 Numerical checks of the asymptotic expansion

Bulirsch-Stoer (BST) algorithm (see *e.g.* [38]) is an extrapolation method for convergence acceleration of algebraically converging sequences q_L , $L \in \mathbb{N}^*$. It assumes that q_L behaves for large L as $p_0 + p_1 L^{-\omega} + p_2 L^{-2\omega} + \dots$ for some exponent $\omega > 0$. Given values q_j , $j = 1, \dots, M$ of the sequence, the algorithm provides an estimation of the limit p_0 , together with an estimation of the error. In the usual case where one does not know the value of the parameter ω , it has to be estimated by trying to minimize the estimation of the error, which requires some educated guesswork. In the case considered in this paper, however, we know that the norm of the eigenstates has an asymptotic expansions in $1/\sqrt{L}$. One can then set from the beginning $\omega = 1/2$ when applying BST algorithm. The convergence of the estimation of p_0 to its exact value is then exponentially fast in the number M of values of the sequence supplied to the algorithm, although the larger M is, the more precision is needed for the q_j 's due to fast propagation of rounding errors. BST algorithm thus allows to check to very high accuracy the asymptotics obtained.

We used BST algorithm in order to check (36) for all 139 first eigenstates with $\sum_{a \in A_0^+} a + \sum_{a \in A^+} a + \sum_{a \in A_0^-} a + \sum_{a \in A^-} a \leq 6$ (giving 57 different values for the norms due to degeneracies). We computed numerically the exact formula (21) in terms of the Bethe roots, divided by all the factors of the asymptotics except the exponential of the integral, for several values of the system size L and fixed density of particles ρ . We then compared the result of BST algorithm for each value of ρ with the numerical value of the exponential of the integral, which was computed by cutting the integral into three pieces: from $-\infty$ to -1 with the integrand $\varphi_r'(u)^2 + 2m_r^2/u$, from -1 to 0 with the integrand $\varphi_r'(u)^2$ and from 0 to $\varphi_r^{-1}(s)$ with the integrand $\varphi_r'(u)^2$. The first piece absorbs the divergence at $u \rightarrow -\infty$, while the last two pieces make sure that the path of integration does not cross the branch cuts of φ_r .

All the computations were done with the generic value $s = 0.2 + i$ for the rescaled fugacity. Solving numerically the equation (23) for b , calculating the Bethe roots from (25), inverting the field φ_r , and evaluating numerically the integrals are relatively costly in computer time, especially with a large number of digits. The exact formula was computed with 200 significant digits, for $\rho = 1/2$ with $L = 12, 14, 16, \dots, 200$, for $\rho = 1/3$ with $L = 18, 21, 24, \dots, 300$, for $\rho = 1/4$ with $L = 24, 28, 32, \dots, 400$, and for $\rho = 1/5$ with $L = 30, 35, 40, \dots, 500$. The estimated relative error from BST algorithm was lower than 10^{-50} in all

cases. Comparing with the numerical evaluation of the integrals, we found a perfect agreement within at least 50 digits in relative accuracy.

3.10 Current fluctuations

From the asymptotics (36) of the norm, we are now in position to write the large L , T limit with

$$T = \frac{t L^{3/2}}{\sqrt{\rho(1-\rho)}} \quad (38)$$

of the generating function for the current fluctuations, in the case of an evolution conditioned on the initial condition \mathcal{C}_X with particles at position $(X, X+1, \dots, X+N-1)$ and on the final condition \mathcal{C}_Y with particles at position $(Y, Y+1, \dots, Y+N-1)$. The distance between X and Y is taken as

$$Y - X = (1 - 2\rho)T + (x + \rho)L. \quad (39)$$

The first term comes from the term of order L^{-1} in the eigenvalue (34) and corresponds to the group velocity, while the second term defines a rescaled distance x on the ring. The current fluctuations are then defined as

$$\xi_{t,x} = \frac{Q_T - JLT}{\sqrt{\rho(1-\rho)L^{3/2}}}, \quad (40)$$

where the mean value of the current per site J has to be set equal to

$$JLT = \rho(1-\rho)LT - \rho(1-\rho)L^2. \quad (41)$$

Its leading term $\rho(1-\rho)$ is the stationary value of the current. The negative correction $-\rho(1-\rho)L^2$ is caused by the beginning and the end of the evolution, during which the particles can not easily move due to the initial and final conditions chosen. Its value follows [39] from Burgers' equation.

Multiplying the norm (36) by $e^{2i\pi p_r(Y-X)/L}$ to change the final state from \mathcal{C}_X to \mathcal{C}_Y , the choices (39) and (41) imply for the generating function of the current fluctuations $G_{t,x}(s) = \langle e^{\gamma(Q_T - JLT)} \rangle = \langle e^{s\xi_{t,x}} \rangle$ the large L , T limit

$$G_{t,x}(s) \rightarrow \frac{1}{Z_t} \sum_r \omega_r^2 \frac{e^{2i\pi p_r x} e^{\varphi_r^{-1}(s)}}{\varphi_r'(\varphi_r^{-1}(s))} \exp \left[\lim_{\Lambda \rightarrow \infty} R^\Lambda + \int_{-\Lambda}^{\varphi_r^{-1}(s)} du \left(\varphi_r'(u)^2 + t \varphi_r(u) \right) \right]. \quad (42)$$

The sum is over the infinitely many first excited states r of $M(\gamma)$, characterized by the four sets of positive half-integers A_0^+ , A_0^- , A^+ , A^- with the equalities on their number of elements $m_r^+ = |A_0^+| = |A^+|$, $m_r^- = |A_0^-| = |A^-|$. The function φ_r is defined in (28). The normalization Z_t is such that the generating function is equal to 1 when the rescaled fugacity $s = 0$. The regularization of the integral at $u \rightarrow -\infty$ is $R^\Lambda = tD^\Lambda - 2m_r^2 \log \Lambda$ with D^Λ given by (35). The combinatorial factor ω_r is equal to

$$\omega_r^2 = \frac{(-1)^{m_r} (\pi^2/4)^{m_r^2}}{(4\pi^2)^{m_r}} \omega(A_0^+)^2 \omega(A_0^-)^2 \omega(A^+)^2 \omega(A^-)^2 \omega(A_0^+, A_0^-)^2 \omega(A^+, A^-)^2, \quad (43)$$

with factors defined in (37) and $m_r = m_r^+ + m_r^-$.

Taking s purely imaginary inside the generating function, the probability distribution P_ξ of the random variable $\xi_{t,x}$ can be extracted by Fourier transform

$$P_\xi(w) = \int_{-\infty}^{\infty} \frac{ds}{2\pi} e^{isw} G_{t,x}(-is). \quad (44)$$

We observe that the integral over s can be nicely replaced by an integral over $d = \varphi_r^{-1}(s)$ on some curve in the complex plane. The Jacobian of this change of variables precisely cancels the denominator $\varphi_r'(\varphi_r^{-1}(s))$ in the generating function.

4 Euler-Maclaurin formula

In order to obtain the large L limit for the normalization of the eigenstates, we need to compute the asymptotics of various sums (and products) with a summation range growing as L and a summand involving the summation index j as j/L . Such asymptotics can be performed using the Euler-Maclaurin formula [40]. It turns out that the sums considered here have various singularities (square root, logarithm and worse) at both ends of the summation range, for which the most naive version of the Euler-Maclaurin formula (45) does not work. We discuss here some adaptations of the Euler-Maclaurin formula to logarithmic singularities (Stirling's formula), non-integer powers (Hurwitz zeta function), and logarithm of a difference of square roots ($\sqrt{\text{Stirling}}$ formula). We begin with one-dimensional sums, and consider then sums on some two dimensional domains necessary to treat the Vandermonde determinant of Bethe roots in (21).

4.1 One-dimensional sums

4.1.1 Functions without singularities

The Euler-Maclaurin formula gives an asymptotic expansion for the difference between a Riemann sum and the corresponding integral. Let M and L be positive integers, $\mu = M/L$ their ratio, and f a function with no singularities in a region which contains the segment $[0, \mu]$. Then, for large M , L with fixed μ , the Euler-Maclaurin formula states that

$$\sum_{j=1}^M f\left(\frac{j+d}{L}\right) \simeq L \left(\int_0^\mu du f(u) \right) + (\mathcal{R}_L f)[\mu, d] - (\mathcal{R}_L f)[0, d], \quad (45)$$

where the remainder term is expressed in terms of the Bernoulli polynomials B_ℓ as

$$(\mathcal{R}_L f)[\mu, d] = \sum_{\ell=1}^{\infty} \frac{B_\ell(d+1) f^{(\ell-1)}(\mu)}{\ell! L^{\ell-1}}. \quad (46)$$

A simple derivation of (45) using Hurwitz zeta function ζ (137) consists in replacing the function f by its Taylor series at 0 in the sum. In order to perform the summation over j at each order in the Taylor series, we use

$$\sum_{j=1}^M (j+d)^\nu = \zeta(-\nu, d+1) - \zeta(-\nu, M+d+1), \quad (47)$$

which follows directly from the definition (137) for $\nu < -1$, and then for all $\nu \neq -1$ by analytic continuation. It gives

$$\sum_{j=1}^M f\left(\frac{j+d}{L}\right) = \sum_{k=0}^{\infty} f^{(k)}(0) \frac{\zeta(-k, d+1) - \zeta(-k, M+d+1)}{k! L^k}. \quad (48)$$

The ζ function whose argument depends on M can be expanded for large $M = \mu L$ using (144). This leads to

$$\sum_{j=1}^M f\left(\frac{j+d}{L}\right) \simeq \sum_{k=0}^{\infty} f^{(k)}(0) \frac{\zeta(-k, d+1)}{k! L^k} + L \sum_{k=0}^{\infty} f^{(k)}(0) \frac{\mu^{k+1}}{(k+1)!} - \sum_{r=0}^{\infty} \frac{\zeta(-r, d+1)}{r! L^r} \sum_{k=0}^{\infty} f^{(k+r)}(0) \frac{\mu^k}{k!}. \quad (49)$$

The second term on the right is the Taylor series at $\mu = 0$ of the integral from 0 to μ of f , while in the last term, we recognize the Taylor expansion at $\mu = 0$ of $f^{(r)}(\mu)$. Expressing the remaining ζ function in terms of Bernoulli polynomials using (141), we arrive at (45).

4.1.2 Logarithmic singularity: Stirling's formula

For functions having a singularity at the origin, (45) can no longer be used, since the derivatives at 0 of the function become infinite. A well known example is Stirling's formula for the Γ function, for which one has a logarithmic singularity. One has the identity

$$\sum_{j=1}^M \log(j+d) = \log \Gamma(M+d+1) - \log \Gamma(d+1), \quad (50)$$

valid for $d+1 \notin \mathbb{R}^-$, where the $\log \Gamma$ function $\log \Gamma(z)$ is defined as the analytic continuation with $\log \Gamma(1) = 0$ of $\log(\Gamma(z))$ to \mathbb{C} minus the branch cut \mathbb{R}^- . Then, Stirling's formula can be stated as the asymptotic expansion

$$\sum_{j=1}^M \log\left(\frac{j+d}{L}\right) \simeq L \left(\int_0^\mu du \log u \right) + (\mathcal{R}_L \log)[\mu, d] + (d + \tfrac{1}{2}) \log L + \log \sqrt{2\pi} - \log \Gamma(d+1). \quad (51)$$

This has the same form as the Euler-Maclaurin formula for a regular function (45), except for the remainder term at 0, which involves the non trivial constant $\log \sqrt{2\pi}$ when $d = 0$. The constant term is analytic in d except for the branch cut of $\log \Gamma$ if $\log \Gamma$ is interpreted as the $\log \Gamma$ function and not the logarithm of the Γ function. We use this prescription in the rest of the paper.

4.1.3 Non-integer power singularity: Hurwitz zeta function

Another example of singularities is non-integer power functions. Using (47) for $\nu \neq -1$ and

$$\sum_{j=1}^M \frac{1}{j+d} = -\frac{\Gamma'(d+1)}{\Gamma(d+1)} + \frac{\Gamma'(M+d+1)}{\Gamma(M+d+1)} \quad (52)$$

for $\nu = -1$, the asymptotics of ζ (144) and of Γ give

$$\sum_{j=1}^M \left(\frac{j+d}{L}\right)^\nu \simeq L \left(\oint_0^\mu du u^\nu \right) + (\mathcal{R}_L(\cdot)^\nu)[\mu, d] + \begin{cases} \frac{\zeta(-\nu, d+1)}{L^\nu} & \nu \neq -1 \\ -L \frac{\Gamma'(\nu+1)}{\Gamma(\nu+1)} & \nu = -1 \end{cases}, \quad (53)$$

where $(\cdot)^\nu$ denotes the function $x \mapsto x^\nu$. The modified integral is equal to

$$\oint_0^\mu du u^\nu = \begin{cases} \frac{\mu^{\nu+1}}{\nu+1} & \nu \neq -1 \\ \log \mu - \log L^{-1} & \nu = -1 \end{cases}. \quad (54)$$

It is equal to the usual, convergent, definition of the integral only in the case $\nu > -1$. Both cases in (53) can be unified by replacing ζ by $\tilde{\zeta}$ defined in (138).

4.1.4 Logarithm of a sum of two square roots: $\sqrt{\text{Stirling}}$ formula

In the calculation of the asymptotics of the normalization of Bethe states, more complicated singularities appear, with functions that depend themselves on L . We define

$$\alpha_\pm(u, q) = \log(\sqrt{u} \pm \sqrt{q}). \quad (55)$$

One has the asymptotic expansion

$$\begin{aligned} \sum_{j=1}^M \alpha_\pm\left(\frac{j+d}{L}, \frac{q}{L}\right) &\simeq L \left(\int_0^\mu du \alpha_\pm(u, q/L) \right) + (\mathcal{R}_L \alpha_\pm(\cdot, q/L))[\mu, d] + \frac{q}{2} - \frac{q \log q}{2} \\ &+ \frac{i\pi(1 \mp 1)q}{2} \operatorname{sgn}(\arg q) + \frac{(d + \frac{1}{2}) \log L}{2} + \frac{\log(2\pi)}{4} - \frac{\log \Gamma(d-q+1)}{2} \\ &\pm \int_0^q du \frac{\zeta(\frac{1}{2}, d+u-q+1)}{2\sqrt{u}}, \end{aligned} \quad (56)$$

where the first integral is equal to

$$\int_0^\mu du \alpha_\pm(u, q) = -\frac{\mu}{2} \pm \sqrt{\mu}\sqrt{q} + \frac{q \log q}{2} + \frac{i\pi(-1 \pm 1)q}{2} \operatorname{sgn}(\arg q) + (\mu - q) \log(\sqrt{\mu} \pm \sqrt{q}) . \quad (57)$$

The argument of q is taken in the interval $(-\pi, \pi)$. The path of integration for the second integral in (56) is required to avoid the branch cuts of the integrand coming from the square root and the ζ function.

The asymptotic expansion (56) is a kind of square root version of Stirling's formula since $\alpha_+(u, q) + \alpha_-(u, q) = \log(u - q)$. Indeed, adding (56) for α_+ and α_- gives (51) after using the property

$$\left(\mathcal{R}_L f\left(\cdot - \frac{q}{L}\right)\right)[\mu, d] = (\mathcal{R}_L f)\left[\mu - \frac{q}{L}, d\right] = (\mathcal{R}_L f)[\mu, d - q] + L \int_{\mu - \frac{q}{L}}^\mu du f(u) , \quad (58)$$

which follows from the relation (142) satisfied by the Bernoulli polynomials.

The expansion (56) is a bit more complicated to show than (51) or (53). It can be derived by using the summation formula

$$\begin{aligned} \sum_{j=1}^M \log(\sqrt{j+d} \pm \sqrt{q}) &= \frac{\log \Gamma(M+d-q+1)}{2} - \frac{\log \Gamma(d-q+1)}{2} \\ &\quad \pm \int_0^q du \frac{\zeta(\frac{1}{2}, d-q+u+1) - \zeta(\frac{1}{2}, M+d-q+u+1)}{2\sqrt{u}} , \end{aligned} \quad (59)$$

which can be proved starting from the identity

$$\partial_\lambda \log(\sqrt{j+d+\lambda} \pm \sqrt{q+\lambda}) = \pm \frac{1}{2\sqrt{j+d+\lambda}\sqrt{q+\lambda}} . \quad (60)$$

Indeed, summing (60) over j using (47) and integrating over λ , there exist a quantity $K_M(d, q)$, independent of λ , such that

$$\sum_{j=1}^M \log(\sqrt{j+d+\lambda} \pm \sqrt{q+\lambda}) = K_M(d, q) \pm \int_0^\lambda du \frac{\zeta(\frac{1}{2}, d+u+1) - \zeta(\frac{1}{2}, M+d+u+1)}{2\sqrt{q+u}} . \quad (61)$$

The constant of integration can be fixed from the special case $\lambda = -q$, using (50). Taking $\lambda = 0$ in the previous equation finally gives (59).

The asymptotic expansion (56) is a consequence of the summation formula (59). Using (47) and (50) we rewrite (59) as

$$\sum_{j=1}^M \log(\sqrt{j+d} \pm \sqrt{q}) = \frac{1}{2} \sum_{j=1}^M \log(j+d-q) \pm \int_0^q \frac{du}{2\sqrt{u}} \sum_{j=1}^M \frac{1}{\sqrt{j+d-q+u}} , \quad (62)$$

where the integration is on a contour that avoids the branch cuts of the square roots. The asymptotic expansions (51) and (53) give

$$\begin{aligned} \sum_{j=1}^M \log\left(\sqrt{\frac{j+d}{L}} \pm \sqrt{\frac{q}{L}}\right) &\simeq \frac{L}{2} \left(\int_0^\mu du \log u \right) + \frac{1}{2}(d-q+\frac{1}{2}) \log L + \frac{\log(2\pi)}{4} - \frac{\log \Gamma(d-q+1)}{2} \\ &\quad \pm \int_0^q du \frac{\zeta(\frac{1}{2}, d-q+u+1)}{2\sqrt{u}} \pm \sqrt{L} \left(\int_0^q \frac{du}{2\sqrt{u}} \right) \left(\int_0^\mu \frac{dv}{\sqrt{v}} \right) \\ &\quad \pm \frac{1}{2\sqrt{L}} \int_0^q \frac{du}{\sqrt{u}} \left(\mathcal{R}_L \frac{1}{\sqrt{\cdot}} \right) [\mu, d-q+u] + \frac{(\mathcal{R}_L \log)[\mu, d-q]}{2} . \end{aligned} \quad (63)$$

The operator \mathcal{R}_L , defined in (46), is linear. From (58), it verifies

$$\int_0^q du h(u) (\mathcal{R}_L f)[\mu, d-q+u] = \left(\mathcal{R}_L \left(\int_0^q du h(u) f\left(\cdot + \frac{u}{L}\right) \right) \right) [\mu, d-q] + L \int_0^q du h(u) \int_\mu^{\mu + \frac{u}{L}} dv f(v) . \quad (64)$$

Applying this property to $h(u) = f(u) = u^{-1/2}$ and using (60) to integrate inside the operator \mathcal{R}_L , one has

$$\begin{aligned} & \pm \frac{1}{2\sqrt{L}} \int_0^q \frac{du}{\sqrt{u}} \left(\mathcal{R}_L \frac{1}{\sqrt{\cdot}} \right) [\mu, d - q + u] + \frac{(\mathcal{R}_L \log)[\mu, d - q]}{2} \\ &= \left(\mathcal{R}_L \log(\sqrt{\cdot} \pm \sqrt{q/L}) \right) [\mu, d] - L \int_0^{q/L} du \log(\sqrt{\mu} \pm \sqrt{u}) . \end{aligned} \quad (65)$$

After some simplifications, we arrive at (56).

4.1.5 Singularities at both ends

In all the cases described so far in this section, we observe that the asymptotic expansion can always be written as

$$\sum_{j=1}^M f\left(\frac{j+d}{L}\right) \simeq L \left(\int_0^\mu du f(u) \right) + (\mathcal{R}_L f)[\mu, d] + (\mathcal{S}_L f)[d] , \quad (66)$$

with some regularization needed when the integral does not converge. This is true in general since one can always decompose the sum from 1 to M as a sum from 1 to εL plus a sum from $\varepsilon L + 1$ to M . For all $\varepsilon > 0$ such that εL is an integer, the asymptotics of the second sum is given by (45) and the limit $\varepsilon \rightarrow 0$ can be written as (66) with a singular part $(\mathcal{S}_L f)[d]$ independent of μ .

Let us now consider the case of a function f , with singularities at both 0 and $\rho = N/L$. The singularities are specified by functions $\mathcal{S}_0 = \mathcal{S}_L f$ and $\mathcal{S}_\rho = \mathcal{S}_L f(\rho - \cdot)$ in (66). Splitting the sum into two parts at $M = \mu L$ leads to

$$\sum_{j=1}^N f\left(\frac{j+d}{L}\right) = \sum_{j=1}^M f\left(\frac{j+d}{L}\right) + \sum_{j=1}^{N-M} f\left(\rho - \frac{j-d-1}{L}\right) . \quad (67)$$

One can use (66) on both parts. From (143), the regular remainder terms at μ cancels: $(\mathcal{R}_L f)[\mu, d] + (\mathcal{R}_L f(\rho - \cdot))[\rho - \mu, -d - 1] = 0$, leaving only the integral and the singular terms:

$$\sum_{j=1}^N f\left(\frac{j+d}{L}\right) \simeq L \left(\int_0^\rho du f(u) \right) + \mathcal{S}_0[d] + \mathcal{S}_\rho[-d - 1] . \quad (68)$$

In particular, for

$$f(x) = \alpha \log x + \sum_{k=-1}^{\infty} f_k x^{k/2} = \bar{\alpha} \log(\rho - x) + \sum_{k=-1}^{\infty} \bar{f}_k (\rho - x)^{k/2} , \quad (69)$$

one has the asymptotic expansion

$$\begin{aligned} \sum_{j=1}^N f\left(\frac{j+d}{L}\right) &\simeq L \left(\int_0^\rho du f(u) \right) + (\alpha - \bar{\alpha})(d + \tfrac{1}{2}) \log L + (\alpha + \bar{\alpha}) \log \sqrt{2\pi} - \alpha \log \Gamma(d + 1) \\ &\quad - \bar{\alpha} \log \Gamma(-d) + \sum_{k=-1}^{\infty} f_k \frac{\zeta(-k/2, d+1)}{L^{k/2}} + \sum_{k=-1}^{\infty} \bar{f}_k \frac{\zeta(-k/2, -d)}{L^{k/2}} . \end{aligned} \quad (70)$$

Similarly let us consider a function f with square root singularities and singularities as a sum of two square roots, both at 0 and ρ :

$$f(x) = \left(\sqrt{x} + \sigma_0 \sqrt{\frac{q_0}{L}} \right) h_0(x) = \left(\sqrt{\rho - x} + \sigma_1 \sqrt{\frac{q_1}{L}} \right) h_1(\rho - x) . \quad (71)$$

The parameters q_0 and q_1 are complex numbers, σ_0 and σ_1 are equal to 1 or -1 . The functions h_0 and h_1 have only square root singularities at 0:

$$h_0(x) = \exp\left(\sum_{r=0}^{\infty} h_{0,r} x^{r/2}\right) \quad \text{and} \quad h_1(x) = \exp\left(\sum_{r=0}^{\infty} h_{1,r} x^{r/2}\right), \quad (72)$$

with coefficients $h_{0,r}$ and $h_{1,r}$ which may depend on L . Using (56) and (53), one finds after some simplifications the asymptotic expansion

$$\begin{aligned} \sum_{j=m_0}^{N-m_1} \log f\left(\frac{j+d}{L}\right) &\simeq L \left(\int_0^\rho du \log f(u) \right) + \frac{m_0 + m_1 - 1}{2} \log L + \log \sqrt{2\pi} \\ &+ \frac{q_0}{2} - \frac{q_0 \log q_0}{2} + \frac{i\pi(1-\sigma_0)q_0}{2} \operatorname{sgn}(\arg q_0) - \frac{\log \Gamma(m_0 + d - q_0)}{2} + \sigma_0 \int_0^{q_0} du \frac{\zeta(\frac{1}{2}, m_0 + d - q_0 + u)}{2\sqrt{u}} \\ &+ \frac{q_1}{2} - \frac{q_1 \log q_1}{2} + \frac{i\pi(1-\sigma_1)q_1}{2} \operatorname{sgn}(\arg q_1) - \frac{\log \Gamma(m_1 - d - q_1)}{2} + \sigma_1 \int_0^{q_1} du \frac{\zeta(\frac{1}{2}, m_1 - d - q_1 + u)}{2\sqrt{u}} \\ &+ \sum_{r=0}^{\infty} \frac{h_{0,r} \zeta(-r/2, m_0 + d)}{L^{r/2}} + \sum_{r=0}^{\infty} \frac{h_{1,r} \zeta(-r/2, m_1 - d)}{L^{r/2}}. \end{aligned} \quad (73)$$

The integers m_0 and m_1 were added in order to treat singularities that may appear for j close to 1 and N such that $f((j+d)/L) = 0$. Their contribution to (73) come from (47) and (59).

We assumed that for large L , the coefficients $h_{0,r}$ and $h_{1,r}$ do not grow too fast when r increases. In this paper, we only use (73) with coefficients $h_{0,r}$ and $h_{1,r}$ that have a finite limit when L goes to ∞ , with an expansion in powers of $1/\sqrt{L}$. Also, we only need the expansion up to order L^0 . One has

$$\begin{aligned} &\sum_{r=0}^{\infty} \frac{h_{0,r} \zeta(-r/2, m_0 + d)}{L^{r/2}} + \sum_{r=0}^{\infty} \frac{h_{1,r} \zeta(-r/2, m_1 - d)}{L^{r/2}} \\ &= \frac{1 - m_0 - m_1}{2} \log L + \left(\frac{1}{2} - m_0 - d\right) \log \frac{\sigma_0 f(0)}{\sqrt{q_0}} + \left(\frac{1}{2} - m_1 + d\right) \log \frac{\sigma_1 f(\rho)}{\sqrt{q_1}} + \mathcal{O}\left(\frac{1}{\sqrt{L}}\right). \end{aligned} \quad (74)$$

4.2 Two-dimensional sums

Generalizations of the Euler-Maclaurin formula can also be used in the case of summations over two indices. Things are however more complicated than in the one-dimensional case because the way to handle those sums depends on the two-dimensional domain of summation, and because of the new kinds of singularities that can happen at singular points of the boundary. We consider here only the case of rectangles $\{(j, j'), 1 \leq j \leq M, 1 \leq j' \leq M'\}$ and triangles $\{(j, j'), 1 \leq j < j' \leq M\}$ that are needed for the asymptotic expansion of the normalization.

4.2.1 Rectangle with square root singularities at a corner

We consider a function of two variables f with square root singularities at the point $(0, 0)$

$$f(u, v) = \sum_{k=0}^{\infty} \sum_{k'=0}^{\infty} f_{k,k'} u^{k/2} v^{k'/2}, \quad (75)$$

and define two auxiliary functions on the edges of the square

$$g_k(v) = \sum_{k'=0}^{\infty} f_{k,k'} v^{k'/2} \quad \text{and} \quad h_{k'}(u) = \sum_{k=0}^{\infty} f_{k,k'} u^{k/2}. \quad (76)$$

Then, taking $M = \mu L$ and $N = \rho L$, (47) gives the large L asymptotic expansion

$$\begin{aligned}
\sum_{j=1}^M \sum_{j'=1}^N f\left(\frac{j+d}{L}, \frac{j'+d'}{L}\right) &\simeq L^2 \int_0^\mu du \int_0^\rho dv f(u, v) \\
&+ \sum_{\ell=1}^\infty \frac{B_\ell(d+1)}{\ell! L^{\ell-2}} \int_0^\rho dv f^{(\ell-1,0)}(\mu, v) + \sum_{\ell=1}^\infty \frac{B_\ell(d'+1)}{\ell! L^{\ell-2}} \int_0^\mu du f^{(0,\ell-1)}(u, \rho) \\
&+ \sum_{k=0}^\infty \frac{\zeta(-k/2, d+1)}{L^{\frac{k}{2}-1}} \int_0^\rho dv g_k(v) + \sum_{k=0}^\infty \frac{\zeta(-k/2, d'+1)}{L^{\frac{k}{2}-1}} \int_0^\mu du h_k(u) \\
&+ \sum_{\ell, \ell'=1}^\infty \frac{B_\ell(d+1)}{\ell! L^{\ell-1}} \frac{B_{\ell'}(d'+1)}{\ell'! L^{\ell'-1}} f^{(\ell-1, \ell'-1)}(\mu, \rho) + \sum_{k=0}^\infty \frac{\zeta(-k/2, d+1)}{L^{\frac{k}{2}}} \sum_{\ell=1}^\infty \frac{B_\ell(d'+1)}{\ell! L^{\ell-1}} g_k^{(\ell-1)}(\rho) \\
&+ \sum_{k=0}^\infty \frac{\zeta(-k/2, d'+1)}{L^{\frac{k}{2}}} \sum_{\ell=1}^\infty \frac{B_\ell(d+1)}{\ell! L^{\ell-1}} h_k^{(\ell-1)}(\mu) + \sum_{k, k'=0}^\infty \frac{\zeta(-k/2, d+1) \zeta(-k'/2, d'+1)}{L^{\frac{k+k'}{2}}} f_{k, k'} .
\end{aligned} \tag{77}$$

The first term with the double integral is related to the full square, the next four terms with a single integral to the four edges of the square, and the four last terms to the four corners of the square.

4.2.2 Triangle with square root singularities at a corner

We consider again a function of two variables f with square root singularities at $(0, 0)$ as in (75), and define

$$f_k(v) = \sum_{k'=0}^\infty f_{k, k'} v^{k'/2} . \tag{78}$$

One has the asymptotic expansion

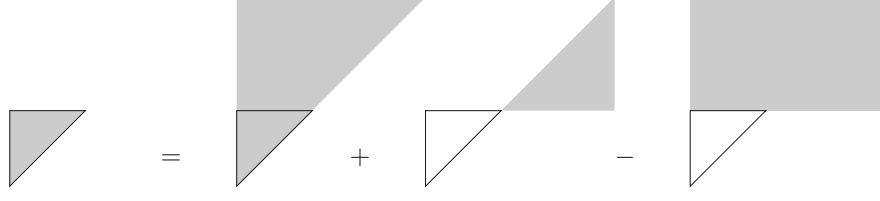
$$\begin{aligned}
\sum_{j=1}^M \sum_{j'=j+1}^M f\left(\frac{j+d}{L}, \frac{j'+d'}{L}\right) &\simeq L^2 \int_0^\mu du \int_u^\mu dv f(u, v) \\
&+ \sum_{\ell=1}^\infty \frac{B_\ell(d'+1)}{\ell! L^{\ell-2}} \partial_\mu^{\ell-1} \int_0^\mu du f(u, \mu) + \sum_{k=0}^\infty \frac{\zeta(-k/2, d+1)}{L^{\frac{k}{2}-1}} \int_0^\mu dv f_k(v) \\
&\quad + \sum_{\ell=1}^\infty \frac{B_\ell(d-d')}{\ell! L^{\ell-2}} \oint_0^\mu du f^{(\ell-1,0)}(u, u) \\
&+ \sum_{\ell, \ell'=1}^\infty \frac{B_\ell(d-d')}{\ell! L^{\ell-1}} \frac{B_{\ell'}(d'+1)}{\ell'! L^{\ell'-1}} \partial_\mu^{\ell-1} f^{(\ell-1,0)}(\mu, \mu) + \sum_{k=0}^\infty \frac{\zeta(-k/2, d+1)}{L^{\frac{k}{2}}} \sum_{\ell=1}^\infty \frac{B_\ell(d'+1)}{\ell! L^{\ell-1}} f_k^{(\ell-1)}(\mu) \\
&\quad + \sum_{k, k'=0}^\infty \frac{f_{k, k'}}{L^{\frac{k+k'}{2}}} \tilde{\zeta}_0(-k/2, -k'/2; d+1, d'+1) .
\end{aligned} \tag{79}$$

The modified integral is defined as in (54), after expanding near $u = 0$. The modified double Hurwitz zeta function $\tilde{\zeta}_0$ is defined in appendix A. The first term in (79) corresponds to the whole triangle, the next three terms to the three edges, and the last three terms to the three corners.

As usual, (79) can be shown by expanding f near the point $(0, 0)$. At each order in the expansion, the summation over j, j' can be performed in terms of double Hurwitz zeta functions using

$$\begin{aligned}
\sum_{j=1}^M \sum_{j'=j+1}^M (j+d)^\nu (j'+d')^{\nu'} &= \zeta(-\nu, -\nu'; d+1, d'+1) \\
&+ \zeta(-\nu', -\nu; M+d'+1, M+d) - \zeta(-\nu, d+1) \zeta(-\nu', M+d'+1) .
\end{aligned} \tag{80}$$

The summation formula (80) can be shown by using the decomposition



$$= \quad , \quad (81)$$

writing

$$\begin{aligned} \sum_{j=1}^M \sum_{j'=j+1}^M (j+d)^\nu (j'+d')^{\nu'} &= \sum_{j=1}^{\infty} \sum_{j'=j+1}^{\infty} (j+d)^\nu (j'+d')^{\nu'} \\ &+ \sum_{j'=M+1}^{\infty} \sum_{j=j'}^{\infty} (j+d)^\nu (j'+d')^{\nu'} - \sum_{j=1}^{\infty} \sum_{j'=M+1}^{\infty} (j+d)^\nu (j'+d')^{\nu'} , \end{aligned} \quad (82)$$

provided that $\nu' < -1$ and $\nu + \nu' < -2$ to ensure the convergence of the infinite sums. From the definition (146) of double Hurwitz zeta functions, this leads to (80). By analytic continuation, (80) is valid for all ν, ν' different from the poles of simple and double ζ . It is also valid when replacing the double ζ by their modified values $\tilde{\zeta}_\alpha$ and $\tilde{\zeta}_{1-\alpha}$ when $2-s-s' \in \mathbb{N}$. Indeed, using (143), we observe that the quantity $\tilde{\zeta}_\alpha(s, s'; z, z') + \tilde{\zeta}_{1-\alpha}(s', s; M+z', M+z-1)$ is not divergent on the line $2-s-s' \in \mathbb{N}$, and is independent of α . One has

$$\lim_{s+s' \rightarrow 2-n} (\zeta(s, s'; z, z') - \zeta(s', s; M+z', M+z-1)) = \tilde{\zeta}_\alpha(s, s'; z, z') - \tilde{\zeta}_{1-\alpha}(s', s; M+z', M+z-1) \quad (83)$$

for arbitrary direction in the convergence of $s+s'$ to $2-n$ and for arbitrary α .

Expanding for large $M = \mu L$ using (144) and (156), a tedious calculation finally leads to the large L asymptotic expansion (79).

4.2.3 Triangle with square root singularities at all corners

We consider a function $f(u, v)$ of two variables, analytic in the interior of the domain $\{(u, v), 0 < u < v < \rho\}$ and with square root singularities at the points $(0, 0), (\rho, \rho), (0, \rho)$. We define $g(u, v) = f(\rho - v, \rho - u)$ and $h(u, v) = f(u, \rho - v)$. The expansions near the singularities are given by

$$f(u, v) = \sum_{k=0}^{\infty} \sum_{k'=0}^{\infty} f_{k,k'} u^{k/2} v^{k'/2} , \quad g(u, v) = \sum_{k=0}^{\infty} \sum_{k'=0}^{\infty} g_{k,k'} u^{k/2} v^{k'/2} , \quad h(u, v) = \sum_{k=0}^{\infty} \sum_{k'=0}^{\infty} h_{k,k'} u^{k/2} v^{k'/2} . \quad (84)$$

We also define

$$f_\nu(v) = \sum_{k'=0}^{\infty} f_{\nu,k'} v^{k'/2} , \quad g_\nu(v) = \sum_{k'=0}^{\infty} g_{\nu,k'} v^{k'/2} . \quad (85)$$

In order to handle the singularities at the corners, we decompose the triangle as

$$\begin{aligned} \sum_{j=1}^N \sum_{j'=j+1}^N f\left(\frac{j+d}{L}, \frac{j'+d'}{L}\right) &= \sum_{j=1}^M \sum_{j'=j+1}^M f\left(\frac{j+d}{L}, \frac{j'+d'}{L}\right) \\ &+ \sum_{j'=1}^{N-M} \sum_{j=j'+1}^{N-M} g\left(\frac{j'-d'-1}{L}, \frac{j-d-1}{L}\right) + \sum_{j=1}^M \sum_{j'=1}^{N-M} h\left(\frac{j+d}{L}, \frac{j'-d'-1}{L}\right) . \end{aligned} \quad (86)$$

Then, for large $N = \rho L$ and $M = \mu L$, using (77) and (79) and combining all the terms leads to the large L asymptotic expansion

$$\begin{aligned}
& \sum_{j=1}^N \sum_{j'=j+1}^N f\left(\frac{j+d}{L}, \frac{j'+d'}{L}\right) \simeq L^2 \int_0^\rho du \int_u^\rho dv f(u, v) \\
& + \sum_{k=0}^\infty \frac{\zeta(-k/2, d+1)}{L^{k/2-1}} \int_0^\rho dv f_k(v) + \sum_{k=0}^\infty \frac{\zeta(-k/2, -d')}{L^{k/2-1}} \int_0^\rho dv g_k(v) \\
& + \sum_{\ell=1}^\infty \frac{B_\ell(d-d')}{\ell! L^{\ell-2}} \left(\oint_0^\mu dv f^{(l-1,0)}(v, v) + \sum_{m=1}^{\ell-1} (-1)^{\ell-m} f^{(m-1, l-m-1)}(\mu, \mu) \right. \\
& \quad \left. + (-1)^{\ell-1} \oint_\mu^\rho dv f^{(0, l-1)}(v, v) \right) \\
& + \sum_{k=0}^\infty \sum_{k'=0}^\infty \frac{f_{k,k'}}{L^{\frac{k+k'}{2}}} \tilde{\zeta}_0(-k/2, -k'/2, d+1, d'+1) + \sum_{k=0}^\infty \sum_{k'=0}^\infty \frac{g_{k',k}}{L^{\frac{k+k'}{2}}} \tilde{\zeta}_0(-k'/2, -k/2, -d', -d) \\
& \quad + \sum_{k=0}^\infty \sum_{k'=0}^\infty \frac{h_{k,k'}}{L^{\frac{k+k'}{2}}} \zeta(-k/2, d+1) \zeta(-k'/2, -d') .
\end{aligned} \tag{87}$$

The modified integral is defined as in (54), after expanding near $v = 0$ and $v = \rho$. The expansion is independent of the arbitrary parameter μ , $0 < \mu < \rho$ that splits the modified integral.

4.2.4 Triangle with logarithmic singularity on an edge: Barnes function

A two dimensional generalization of Stirling's formula for the Γ function is

$$\begin{aligned}
& \sum_{j=1}^N \sum_{j'=j+1}^N \log(j' - j + d) \simeq \frac{N^2 \log N}{2} - \frac{3N^2}{4} + d N \log N \\
& + N \left(\log \sqrt{2\pi} - d - \log \Gamma(d+1) \right) + \left(\frac{d^2}{2} - \frac{1}{12} \right) \log N + \zeta'(-1) + d \log \sqrt{2\pi} - \log G(d+1) ,
\end{aligned} \tag{88}$$

where ζ is Riemann's zeta function and $\log G$ is the log Barnes function, equal to the analytic continuation of the logarithm of the Barnes function G . The constant term is usually written in terms of the Glaisher-Kinkelin constant $A = \exp(\frac{1}{12} - \zeta'(-1))$. The expansion (88) follows from the identity $G(z+1) = \Gamma(z)G(z)$ and the asymptotics of Barnes function for large argument

$$\log G(N+1) \simeq \frac{N^2 \log N}{2} - \frac{3N^2}{4} + \frac{\log(2\pi)}{2} N - \frac{\log N}{12} + \zeta'(-1) . \tag{89}$$

4.2.5 Square with logarithm of a sum of square roots

We consider two dimensional generalizations of the $\sqrt{\text{Stirling}}$ formula (56). One has the asymptotics

$$\begin{aligned}
& \sum_{j=1}^N \sum_{j'=1}^N \log(\sqrt{j+d} + \sqrt{j'+d}) \simeq \frac{N^2 \log N}{2} + \frac{N^2}{4} + (2d+1)N \\
& + 4\sqrt{N} \zeta(-\frac{1}{2}, d+1) - \frac{\log N}{24} + (d + \frac{1}{2})^2 + \int_0^{d+\frac{1}{2}} du \frac{\zeta(\frac{1}{2}, \frac{1}{2} + u)^2}{2} + \kappa_0 ,
\end{aligned} \tag{90}$$

with $\kappa_0 \approx -0.128121307412384$. In order to show this, we introduce $M = \mu N$, $0 < \mu < 1$ and decompose the sum as

$$\sum_{j=1}^N \sum_{j'=1}^N = \sum_{j=1}^M \sum_{j'=1}^M + \sum_{j=M+1}^N \sum_{j'=1}^M + \sum_{j=1}^M \sum_{j'=M+1}^N + \sum_{j=M+1}^N \sum_{j'=M+1}^N . \tag{91}$$

The last three terms in the right hand side can be evaluated using Euler-Maclaurin formula in a rectangle with only square root singularities, using (77) and

$$\begin{aligned} & \log(\sqrt{u+\alpha} + \sqrt{v+\beta}) \\ &= \partial_u \left(-\frac{u}{2} + \sqrt{u+\alpha}\sqrt{v+\beta} + (u+\alpha-v-\beta) \log(\sqrt{u+\alpha} + \sqrt{v+\beta}) \right) \\ &= \partial_u \partial_v \left(-\frac{3uv}{4} + \frac{(u+\alpha)^{3/2}\sqrt{v+\beta}}{2} + \frac{\sqrt{u+\alpha}(v+\beta)^{3/2}}{2} - \frac{(u+\alpha-v-\beta)^2}{2} \log(\sqrt{u+\alpha} + \sqrt{v+\beta}) \right) \end{aligned} \quad (92)$$

to compute the integrals. One finds up to order 0 in N

$$\begin{aligned} & \sum_{j=1}^N \sum_{j'=1}^N \log(\sqrt{j+d} + \sqrt{j'+d'}) - \sum_{j=1}^M \sum_{j'=1}^M \log(\sqrt{j+d} + \sqrt{j'+d'}) \\ & \simeq \left(\frac{1}{4} - \frac{\mu^2}{4} - \frac{\mu^2 \log \mu}{2} \right) N^2 + (2d+1)(1-\mu)N + 4(1-\sqrt{\mu})\zeta(-\frac{1}{2}, d+1)\sqrt{N} + \frac{\log \mu}{24}. \end{aligned} \quad (93)$$

The limit $\mu \rightarrow 0$ leads to the divergent terms in (90). The term at order N^0 follows from the summation formula (47) applied to the identity

$$\partial_\lambda \log(\sqrt{j+d+\lambda} + \sqrt{j'+d+\lambda}) = \frac{1}{2\sqrt{j+d+\lambda}\sqrt{j'+d+\lambda}}. \quad (94)$$

The remaining constant of integration κ_0 can be evaluated numerically with high precision using BST algorithm, as described in section 3.9.

4.2.6 Square with logarithm of a sum of square roots (2)

One has the asymptotics

$$\begin{aligned} & \sum_{j=1}^N \sum_{j'=1}^N \log(\sqrt{-i(j+d)} + \sqrt{i(j'+d')}) \simeq \frac{N^2 \log N}{2} + \left(-\frac{3}{4} + \log 2 \right) N^2 \\ & + ((d+d'+1) \log 2 - i(d-d'))N - 2i\sqrt{N}(\zeta(-\frac{1}{2}, d+1) - \zeta(-\frac{1}{2}, d'+1)) + \left(\frac{1}{24} - \frac{(d+d'+1)^2}{4} \right) \log N \\ & - \frac{i(d-d')(d+d'+1)}{2} + \int_{d'+\frac{1}{2}}^0 du \frac{\zeta(\frac{1}{2}, d+1+u)\zeta(\frac{1}{2}, d'+1-u)}{2i} + \kappa_1(d+d'). \end{aligned} \quad (95)$$

When $d+d' = -1$, the constant of integration is $\kappa_1(-1) \approx 0.05382943932689441$. The derivation is essentially identical to the one of (90).

5 Large L asymptotics

In this section, we compute the large L asymptotics of the quantities

$$\Xi_1 = \frac{1}{N} \sum_{j=1}^N \frac{y_j}{\rho + (1-\rho)y_j}, \quad (96)$$

$$\Xi_2 = \prod_{j=1}^N \left(1 + \frac{1-\rho}{\rho} y_j \right), \quad (97)$$

$$\Xi_3 = \prod_{j=1}^N \prod_{k=j+1}^N \frac{y_j - y_k}{y_j^0 - y_k^0}, \quad (98)$$

$$\Xi_4 = \prod_{j=1}^N \prod_{k=j+1}^N (y_j^0 - y_k^0), \quad (99)$$

for y_j 's given by (25), k_j 's constructed in terms of sets A_0^\pm , A^\pm , and with the correction (32) to b . The parameter c will be in this section an arbitrary complex number that is **not** required to verify (33).

5.1 Function Φ

We introduce the function Φ defined from g (22) by

$$\Phi(u) = g^{-1}(e^{-b_0+2i\pi u}) . \quad (100)$$

Writing the parameter b as in (32), the Bethe roots can be expressed as

$$y_j = \Phi\left(\frac{k_j + ic}{L}\right) . \quad (101)$$

In particular, for the stationary eigenstate, one has

$$y_j^0 = \Phi\left(-\frac{\rho}{2} + \frac{j - \frac{1}{2} + ic}{L}\right) . \quad (102)$$

For the first eigenstates, the k_j 's added ($\pm(N/2 + a)$, $a \in A^\pm$) correspond to $y_j = \Phi(\pm\frac{\rho}{2} + \frac{i(c \mp ia)}{L})$, while the k_j 's removed ($\pm(N/2 - a)$, $a \in A_0^\pm$) correspond to $y_j = \Phi(\pm\frac{\rho}{2} + \frac{i(c \pm ia)}{L})$. These expressions are suited for the use of the Euler-Maclaurin formula to compute the asymptotics of sums of the form $\sum_{j=1}^N f(y_j)$ and $\sum_{j=1}^N \sum_{j'=j+1}^N f(y_j, y_{j'})$.

The function Φ verifies $\Phi(\pm\rho/2) = -\frac{\rho}{1-\rho}$. The points $\pm\rho/2$ are branch points of the function Φ . The expansion of Φ around them is given by

$$\begin{aligned} \Phi\left(\pm\left(\frac{\rho}{2} - u\right)\right) \simeq & -\frac{\rho}{1-\rho} \left(1 - \frac{\sqrt{2}(1 \mp i)\sqrt{\pi}\sqrt{u}}{\sqrt{\rho(1-\rho)}} \mp \frac{4i\pi(1+\rho)u}{3\rho(1-\rho)} \right. \\ & \left. + \frac{\sqrt{2}(1 \pm i)\pi^{3/2}(1+11\rho+\rho^2)u^{3/2}}{9(\rho(1-\rho))^{3/2}} + \frac{8\pi^2(1+\rho)(1-25\rho+\rho^2)u^2}{135\rho^2(1-\rho)^2} \right) . \end{aligned} \quad (103)$$

A useful property of the function Φ is that its derivative can be expressed in terms of Φ alone. One has

$$\Phi'(u) = -2i\pi \frac{\Phi(u)(1-\Phi(u))}{\rho + (1-\rho)\Phi(u)} . \quad (104)$$

5.2 Asymptotics of Ξ_1

The Bethe roots y_j can be replaced by y_j^0 in (96), up to corrections obtained by summing over the sets A_0^\pm , A^\pm . Writing b as (32), the summand is equal to $y_j/(\rho + (1-\rho)y_j) = f(\rho/2 + (k_j + ic)/L)$ with $f(u) = \Phi(u - \frac{\rho}{2})/(\rho + (1-\rho)\Phi(u - \frac{\rho}{2}))$. One has

$$\begin{aligned} \sum_{j=1}^N \frac{y_j}{\rho + (1-\rho)y_j} = & \sum_{j=1}^N f\left(\frac{j - \frac{1}{2} + ic}{L}\right) + \sum_{a \in A^-} f\left(\frac{i(c+ia)}{L}\right) - \sum_{a \in A_0^-} f\left(\frac{i(c-ia)}{L}\right) \\ & + \sum_{a \in A^+} f\left(\rho + \frac{i(c-ia)}{L}\right) - \sum_{a \in A_0^+} f\left(\rho + \frac{i(c+ia)}{L}\right) . \end{aligned} \quad (105)$$

The function f verifies (69) with first coefficients equal to $\alpha = \bar{\alpha} = 0$ and

$$f_{-1} = -\frac{(1-i)\sqrt{\rho}}{2^{3/2}\sqrt{\pi}\sqrt{1-\rho}} \quad \text{and} \quad \bar{f}_{-1} = -\frac{(1+i)\sqrt{\rho}}{2^{3/2}\sqrt{\pi}\sqrt{1-\rho}} . \quad (106)$$

From (70), the expansion up to order \sqrt{L} of the sum over j in the right hand side of (105) is

$$\sum_{j=1}^N f\left(\frac{j - \frac{1}{2} + ic}{L}\right) \simeq L \left(\int_0^\rho du f(u) \right) + \sqrt{L} (f_{-1} \zeta(\frac{1}{2}, \frac{1}{2} + ic) + \bar{f}_{-1} \zeta(\frac{1}{2}, \frac{1}{2} - ic)) . \quad (107)$$

The integral can be computed by making the change of variables $z = \Phi(u - \frac{\rho}{2})$, as explained in appendix B. The residue calculation gives $\int_0^\rho du f(u) = 0$.

At leading order in L , using (103) to treat the sums over a , and (140), we can express Ξ_1 at leading order in terms of the derivative of the function φ_r (28). We find

$$\frac{1}{N} \sum_{j=1}^N \frac{y_j}{\rho + (1 - \rho)y_j} \simeq \frac{\varphi'_r(2\pi c)}{\sqrt{\rho(1 - \rho)} \sqrt{L}} . \quad (108)$$

5.3 Asymptotics of Ξ_2

We consider the logarithm of Ξ_2 , defined in (97). The calculation of the large L asymptotics follows closely the one for Ξ_1 , except one needs to push the expansion of the sum up to the constant term in L in order to get the prefactor of the exponential in Ξ_2 . The summand is equal to $\log(1 + y_j(1 - \rho)/\rho) = f(\rho/2 + (k_j + ic)/L)$ with $f(u) = \log(1 + \Phi(u - \frac{\rho}{2})(1 - \rho)/\rho)$. One has again

$$\begin{aligned} \sum_{j=1}^N \log\left(1 + \frac{1 - \rho}{\rho} y_j\right) &= \sum_{j=1}^N f\left(\frac{j - \frac{1}{2} + ic}{L}\right) + \sum_{a \in A^-} f\left(\frac{i(c + ia)}{L}\right) - \sum_{a \in A_0^-} f\left(\frac{i(c - ia)}{L}\right) \\ &\quad + \sum_{a \in A^+} f\left(\rho + \frac{i(c - ia)}{L}\right) - \sum_{a \in A_0^+} f\left(\rho + \frac{i(c + ia)}{L}\right) . \end{aligned} \quad (109)$$

The function f verifies (69) with first coefficients equal to $\alpha = \bar{\alpha} = 1/2$, $f_{-1} = \bar{f}_{-1} = 0$ and

$$f_0 = \log \frac{(1 + i)\sqrt{2\pi}}{\sqrt{\rho(1 - \rho)}} \quad \text{and} \quad \bar{f}_0 = \log \frac{(1 - i)\sqrt{2\pi}}{\sqrt{\rho(1 - \rho)}} . \quad (110)$$

From (70), the expansion up to order L^0 of the sum over j in the right hand side of (109) is

$$\begin{aligned} \sum_{j=1}^N f\left(\frac{j - \frac{1}{2} + ic}{L}\right) &\simeq L \left(\int_0^\rho du f(u) \right) + \log(2\pi) - \frac{\log \Gamma(\frac{1}{2} + ic)}{2} - \frac{\log \Gamma(\frac{1}{2} - ic)}{2} \\ &\quad + f_0 \zeta(0, \frac{1}{2} + ic) + \bar{f}_0 \zeta(0, \frac{1}{2} - ic) . \end{aligned} \quad (111)$$

Again, the integral can be computed as explained in appendix B. One finds $\int_0^\rho du f(u) = 0$.

At leading order in L , using (103) and the constraint (27) to treat the sums over a , and $\zeta(0, z) = \frac{1}{2} - z$ (141), one can express the expansion of $\log \Xi_2$ to order L^0 . After taking the exponential and using Euler's reflection formula $\Gamma(z)\Gamma(1 - z) = \pi/\sin(\pi z)$ to eliminate the Γ functions, we obtain

$$\prod_{j=1}^N \left(1 + \frac{1 - \rho}{\rho} y_j\right) \simeq \sqrt{1 + e^{2\pi c}} \frac{(\prod_{a \in A^+} \sqrt{c - ia})(\prod_{a \in A^-} \sqrt{c + ia})}{(\prod_{a \in A_0^+} \sqrt{c + ia})(\prod_{a \in A_0^-} \sqrt{c - ia})} . \quad (112)$$

The factor $\sqrt{1 + e^{2\pi c}}$ has infinitely many branch cuts $i(n + \frac{1}{2}) + \mathbb{R}^+$, $n \in \mathbb{Z}$. Since $\log \Gamma$ is interpreted as the log Γ function and not the logarithm of the Γ function in (111), $\sqrt{1 + e^{2\pi c}}$ has to be understood for $\text{Re } c > 0$ as the analytic continuation in c from the real axis $(-1)^{\lfloor \frac{\text{Im}(4\pi c + 2i\pi)}{4\pi} \rfloor} \sqrt{1 + e^{2\pi c}}$ with $\lfloor x \rfloor$ the largest integer smaller or equal to x . This corresponds to choosing instead the branch cuts $(-\infty, -i/2] \cup [i/2, \infty)$ for $\sqrt{1 + e^{2\pi c}}$.

5.4 Asymptotics of Ξ_3

We consider the quantity Ξ_3 , defined in (98). For the stationary state, one has $\Xi_3 = 1$. We focus on the other first eigenstates, and take the parameter c with $\text{Re } c > 0$: this constraint is verified for the solution of (33) as long as the real part of s is not too negative. In particular, it seems to be valid for all s with $\text{Re } s \geq 0$. This is not the case for the stationary state, for which the solution of (33) at leading order in L is $c \rightarrow -\infty$ when $s \rightarrow 0$.

Replacing y_j and y_k by y_j^0 and y_k^0 in the definition (98) of Ξ_3 , with corrections coming from the sets A_0^\pm , A^\pm , one has

$$\prod_{j=1}^N \prod_{k=j+1}^N \frac{y_j - y_k}{y_j^0 - y_k^0} = (\text{finite products}) \quad (113)$$

$$\times \frac{\prod_{a \in A^+} \prod_{j=1}^N \left(\Phi\left(-\frac{\rho}{2} + \frac{j-\frac{1}{2}+ic}{L}\right) - \Phi\left(\frac{\rho}{2} + \frac{a+ic}{L}\right) \right)}{\prod_{a \in A_0^+} \prod_{\substack{j=1 \\ n-j \neq a-\frac{1}{2}}}^N \left(\Phi\left(-\frac{\rho}{2} + \frac{j-\frac{1}{2}+ic}{L}\right) - \Phi\left(\frac{\rho}{2} - \frac{a-ic}{L}\right) \right)}$$

$$\times \frac{\prod_{a \in A^-} \prod_{j=1}^N \left(\Phi\left(-\frac{\rho}{2} + \frac{j-\frac{1}{2}+ic}{L}\right) - \Phi\left(-\frac{\rho}{2} - \frac{a-ic}{L}\right) \right)}{\prod_{a \in A_0^-} \prod_{\substack{j=1 \\ j \neq a+\frac{1}{2}}}^N \left(\Phi\left(-\frac{\rho}{2} + \frac{j-\frac{1}{2}+ic}{L}\right) - \Phi\left(-\frac{\rho}{2} + \frac{a+ic}{L}\right) \right)}.$$

The (finite products) factor contains all the factors with only contributions from the sets A_0^\pm , A^\pm and no product over j between 1 and N . Their asymptotics can be computed from (103) and (27). At leading order in L , one finds

$$(\text{finite products}) \simeq \left(\frac{(1-\rho)^{3/2}}{2i\sqrt{\pi}\sqrt{\rho}} \sqrt{L} \right)^{m_r} \left(\prod_{a \in A_0^+} (-1)^{a-\frac{1}{2}} \right) \left(\prod_{a \in A_0^-} (-1)^{a+\frac{1}{2}} \right) \quad (114)$$

$$\times \prod_{\sigma \in \{-, +\}} \frac{\prod_{\substack{a, a' \in A^\sigma \\ \sigma a < \sigma a'}} \left(\sqrt{c - \sigma i a} - \sqrt{c - \sigma i a'} \right) \prod_{\substack{a, a' \in A_0^\sigma \\ \sigma a > \sigma a'}} \left(\sqrt{c + \sigma i a} - \sqrt{c + \sigma i a'} \right)}{\prod_{a \in A^\sigma} \prod_{a' \in A_0^\sigma} \left(\sqrt{c - \sigma i a} - \sqrt{c + \sigma i a'} \right)}$$

$$\times \frac{\prod_{a \in A^+} \prod_{a' \in A^-} \left(\sqrt{c - i a} + \sqrt{c + i a'} \right) \prod_{a \in A_0^+} \prod_{a' \in A_0^-} \left(\sqrt{c + i a} + \sqrt{c - i a'} \right)}{\prod_{a \in A^+} \prod_{a' \in A_0^-} \left(\sqrt{c - i a} + \sqrt{c - i a'} \right) \prod_{a \in A_0^+} \prod_{a' \in A^-} \left(\sqrt{c + i a} + \sqrt{c + i a'} \right)}.$$

After taking the logarithm, the factors that contain a product over j in (113) take the form of a sum for j between 1 and N of $\log f((j - \frac{1}{2} + ic)/L)$ where

$$f(x) = \Phi\left(-\frac{\rho}{2} + x\right) - \Phi\left(\frac{\sigma\rho}{2} + \frac{ic + \sigma' a}{L}\right), \quad (115)$$

with parameters, $\sigma, \sigma' \in \{+1, -1\}$, $a \in \mathbb{N} + \frac{1}{2}$.

The function f has a singularity when either x or $\rho - x$ is of order $1/L$: using (103) and the assumption $\operatorname{Re} c > 0$, one has for large L

$$\begin{aligned} f(x/L) &\simeq \frac{\sqrt{2}(1+i)\sqrt{\pi}\sqrt{\rho}}{(1-\rho)^{3/2}} \left(\sqrt{\frac{x}{L}} + \sigma \sqrt{\frac{ic + \sigma'a}{L}} \right) \\ f(\rho - x/L) &\simeq \frac{\sqrt{2}(1-i)\sqrt{\pi}\sqrt{\rho}}{(1-\rho)^{3/2}} \left(\sqrt{\frac{x}{L}} - \sigma \sqrt{-\frac{ic + \sigma'a}{L}} \right). \end{aligned} \quad (116)$$

This is precisely the type of singularities that can be treated by (56). Since both ends of the summation range exhibit this singularity, one can directly use (73), with coefficients $\sigma_0 = \sigma$, $\sigma_1 = -\sigma$, $q_0 = ic + \sigma'a$, $q_1 = -ic - \sigma'a$. Introducing nonnegative integers m and m' to avoid terms in the sum for which $f((j - \frac{1}{2} + ic)/L) = 0$, we find the large L asymptotics

$$\begin{aligned} \sum_{j=m}^{N-m'} \log f\left(\frac{j - \frac{1}{2} + ic}{L}\right) &\simeq \rho \log\left(\frac{\rho}{1-\rho}\right)L + \sigma \frac{2i\sqrt{\pi}\sqrt{\rho}\sqrt{c - \sigma'ia}}{\sqrt{1-\rho}} \sqrt{L} + \frac{m+m'-1}{2} \log L \\ &+ \log \sqrt{2\pi} + \frac{2\pi(1-2\rho)c}{3(1-\rho)} - \frac{\pi(1-5\rho)\sigma'ia}{6(1-\rho)} + \frac{i\pi(1-m+m')}{4} \\ &- (m+m'-1) \log \frac{2\sqrt{\pi}\sqrt{\rho}}{(1-\rho)^{3/2}} - \frac{\log \Gamma(m - \frac{1}{2} - \sigma'a)}{2} - \frac{\log \Gamma(m' + \frac{1}{2} + \sigma'a)}{2} \\ &+ \sigma \int_{\sigma'ia}^c du \frac{e^{i\pi/4}\zeta(\frac{1}{2}, m - \frac{1}{2} + iu) - e^{-i\pi/4}\zeta(\frac{1}{2}, m' + \frac{1}{2} - iu)}{2\sqrt{u - \sigma'ia}}. \end{aligned} \quad (117)$$

There, the integral giving the leading order in L was computed as explained at the beginning of appendix B by making the change of variable $z = \Phi(u - \frac{\rho}{2})$. One has

$$\begin{aligned} \int_0^\rho du \log f(u) &= \rho \log \left(-\Phi\left(\frac{\sigma\rho}{2} + \frac{ic + \sigma'a}{L}\right) \right) \\ &\simeq \rho \log \frac{\rho}{1-\rho} + \sigma \frac{2i\sqrt{\pi}\sqrt{\rho}\sqrt{c - \sigma'ia}}{\sqrt{1-\rho}\sqrt{L}} + \frac{2\pi(1-2\rho)(c - \sigma'ia)}{3(1-\rho)L}. \end{aligned} \quad (118)$$

We write \sum' for the full sum between 1 and N minus any divergent term as in (113). Its expansion up to order 0 in L is

$$\begin{aligned} \sum_{j=1}^N \log f\left(\frac{j - \frac{1}{2} + ic}{L}\right) &\simeq \rho \log\left(\frac{\rho}{1-\rho}\right)L + \sigma \frac{2i\sqrt{\pi}\sqrt{\rho}\sqrt{c - \sigma'ia}}{\sqrt{1-\rho}} \sqrt{L} + \frac{1 - \sigma\sigma'}{4} \log L \\ &+ \log \sqrt{2\pi} + \frac{2\pi(1-2\rho)c}{3(1-\rho)} - \frac{\pi(1-5\rho)\sigma'ia}{6(1-\rho)} + \frac{i\pi(\sigma - \sigma')}{8} \\ &- \frac{1 - \sigma\sigma'}{2} \log \frac{2\sqrt{\pi}\sqrt{\rho}}{(1-\rho)^{3/2}} - \frac{\log \Gamma(m - \frac{1}{2} - \sigma'a)}{2} - \frac{\log \Gamma(m' + \frac{1}{2} + \sigma'a)}{2} \\ &+ \sigma \int_{\sigma'ia}^c du \frac{e^{i\pi/4}\zeta(\frac{1}{2}, m - \frac{1}{2} + iu) - e^{-i\pi/4}\zeta(\frac{1}{2}, m' + \frac{1}{2} - iu)}{2\sqrt{u - \sigma'ia}} \\ &+ \sum_{j=1}^{m-1} \log \left(\sqrt{j - \frac{1}{2} + ic} + \sigma \sqrt{ic + \sigma'a} \right) + \sum_{j=1}^{m'} \log \left(\sqrt{j - \frac{1}{2} - ic} - \sigma \sqrt{-ic - \sigma'a} \right). \end{aligned} \quad (119)$$

The integers m and m' must be taken large enough so that the arguments of the Γ functions do not belong to $-\mathbb{N}$. They are also needed to ensure the convergence of the integral at $u = \sigma'ia$, since $\zeta(\frac{1}{2}, -n + \varepsilon) \sim \varepsilon^{-1/2}$ for $n \in \mathbb{N}$. The branch cuts of the integrand as a function of u are chosen equal to $(i\infty, i(m - \frac{1}{2})]$, $(-i\infty, -i(m' + \frac{1}{2})]$ and $(-\infty, \sigma'ia]$.

Since the left hand side of (119) is independent of m, m' , one would like to eliminate them also in the right hand side. This can be done using the relation

$$\begin{aligned}
& \int_{-\Lambda}^q du \frac{e^{i\pi/4}\zeta(\frac{1}{2}, m - \frac{1}{2} + iu) - e^{-i\pi/4}\zeta(\frac{1}{2}, m' + \frac{1}{2} - iu)}{2\sqrt{u - \sigma'ia}} \\
&= \int_{-\Lambda}^q du \frac{e^{i\pi/4}\zeta(\frac{1}{2}, \frac{1}{2} + iu) - e^{-i\pi/4}\zeta(\frac{1}{2}, \frac{1}{2} - iu)}{2\sqrt{u - \sigma'ia}} - \frac{i\pi(m + m' - 1)}{4} \\
&- \sum_{j=1}^{m-1} \left(\log \left(\sqrt{j - \frac{1}{2} + iq} + \sqrt{\sigma'a + iq} \right) - \log \left(\sqrt{\Lambda + i(j - \frac{1}{2})} + \sigma'\sqrt{\Lambda + \sigma'ia} \right) \right) \\
&+ \sum_{j=1}^{m'} \left(\log \left(\sqrt{j - \frac{1}{2} - iq} + \sqrt{-\sigma'a - iq} \right) - \log \left(\sqrt{\Lambda - i(j - \frac{1}{2})} - \sigma'\sqrt{\Lambda + \sigma'ia} \right) \right),
\end{aligned} \tag{120}$$

which follows from (59) and is valid when $\text{Re } q > 0$ and $\text{Re } \Lambda > 0$ with $|\text{Im } \Lambda| < a$ if the path of integration is required to avoid all branch cuts.

We decompose the integral from $\sigma'ia$ to c in (119) as an integral from $-\Lambda$ to c minus the limit $\varepsilon \rightarrow 0$, $\varepsilon > 0$ of an integral from $-\Lambda$ to $\sigma'ia + \varepsilon$. After using (120) for $q = c$ and $q = \sigma'ia + \varepsilon$ the limit $\Lambda \rightarrow \infty$ of the integrals become convergent. The limit $\varepsilon \rightarrow 0$ of the integral between $-\infty$ and $\sigma'ia + \varepsilon$ (with a contour of integration that avoids the branch cut of ζ) is given by the identity

$$\int_{-\infty}^{\sigma'ia+\varepsilon} du \frac{e^{i\pi/4}\zeta(\frac{1}{2}, \frac{1}{2} + iu) - e^{-i\pi/4}\zeta(\frac{1}{2}, \frac{1}{2} - iu)}{2\sqrt{u - \sigma'ia}} \underset{\varepsilon \rightarrow 0}{\simeq} \sigma' \log \sqrt{\varepsilon} + \sigma' \log \sqrt{8\pi} + i\pi a, \tag{121}$$

that was obtained numerically. The divergent contribution $\log \sqrt{\varepsilon}$ cancels with terms in the sums of (120) at $j = a + \frac{1}{2}$. After several simplifications, (119) becomes

$$\begin{aligned}
& \sum_{j=1}^N \log f\left(\frac{j - \frac{1}{2} + ic}{L}\right) \simeq \rho \log\left(\frac{\rho}{1-\rho}\right)L + \sigma \frac{2i\sqrt{\pi}\sqrt{\rho}\sqrt{c - \sigma'ia}}{\sqrt{1-\rho}} \sqrt{L} + \frac{1 - \sigma\sigma'}{4} \log L \\
& - \frac{1 - \sigma\sigma'}{2} \log \left(\frac{\sqrt{\rho}}{2\sqrt{\pi}(1-\rho)^{3/2}\sqrt{c - \sigma'ia}} \right) + \frac{2\pi(1-2\rho)(c - \sigma'ia)}{3(1-\rho)} \\
& - (\sigma - \sigma')i\pi(a - \frac{1}{4}) + \sigma \int_{-\infty}^c du \frac{e^{i\pi/4}\zeta(\frac{1}{2}, \frac{1}{2} + iu) - e^{-i\pi/4}\zeta(\frac{1}{2}, \frac{1}{2} - iu)}{2\sqrt{u - \sigma'ia}}.
\end{aligned} \tag{122}$$

Putting everything together, we obtain for the product of the four factors of Ξ_3 containing products over j in (113)

$$\begin{aligned}
& e^{-\frac{2i\sqrt{\pi}\sqrt{\rho}}{\sqrt{1-\rho}} \left(\sum_{a \in A_0^+} \sqrt{c+ia} + \sum_{a \in A^-} \sqrt{c+ia} - \sum_{a \in A_0^-} \sqrt{c-ia} - \sum_{a \in A^+} \sqrt{c-ia} \right) \sqrt{L}} \\
& \times \left(\frac{\sqrt{\rho}}{2\sqrt{\pi}(1-\rho)^{3/2}\sqrt{L}} \right)^{m_r} i^{m_r^+ - m_r^-} e^{-\frac{2i\pi(1-2\rho)p_r}{3(1-\rho)}} \left(\prod_{a \in A_0^+} \frac{1}{\sqrt{c+ia}} \right) \left(\prod_{a \in A_0^-} \frac{1}{\sqrt{c-ia}} \right) \\
& \times \exp \left[-\frac{1}{2} \int_{-\infty}^c du \left(e^{i\pi/4}\zeta(\frac{1}{2}, \frac{1}{2} + iu) - e^{-i\pi/4}\zeta(\frac{1}{2}, \frac{1}{2} - iu) \right) \right. \\
& \quad \left. \times \left(\sum_{a \in A_0^+} \frac{1}{\sqrt{u+ia}} + \sum_{a \in A^-} \frac{1}{\sqrt{u+ia}} - \sum_{a \in A_0^-} \frac{1}{\sqrt{u-ia}} - \sum_{a \in A^+} \frac{1}{\sqrt{u-ia}} \right) \right].
\end{aligned} \tag{123}$$

5.5 Asymptotics of Ξ_4

We write

$$\prod_{j=1}^N \prod_{j'=j+1}^N (y_j^0 - y_{j'}^0) = i^{\frac{N(N-1)}{2}} \exp \left(\sum_{j=1}^N \sum_{j'=j+1}^N \log(-i(y_j^0 - y_{j'}^0)) \right), \quad (124)$$

where \log is the usual determination of the logarithm with branch cut \mathbb{R}^- . The (clockwise) contour on which the y_j^0 's condense in the complex plane is represented in figure 2. The factor $-i$ in the logarithm ensures that the branch cut of the logarithm is not crossed.

Since $y_j^0 \rightarrow -\rho/(1-\rho)$ when $j \ll L$ and when $N-j \ll L$, the quantity $y_j^0 - y_{j'}^0$ goes to 0 at the three corners of the triangle $\{(j, j'), 1 \leq j < j' \leq N\}$. To avoid extra singularities and allow us to use (87), we consider first the regular part

$$S_{\text{reg}} = \sum_{j=1}^N \sum_{j'=j+1}^N f\left(\frac{j - \frac{1}{2} + ic}{L}, \frac{j' - \frac{1}{2} + ic}{L}\right) \quad (125)$$

with f defined by

$$f(u, v) = \log \left(-i \left(\Phi\left(-\frac{\rho}{2} + u\right) - \Phi\left(-\frac{\rho}{2} + v\right) \right) \frac{(\sqrt{u} + \sqrt{v})(\sqrt{\rho - u} + \sqrt{\rho - v})}{(v - u)(\sqrt{-iu} + \sqrt{i(\rho - v)})} \right). \quad (126)$$

Using the same notations as in (87), one has

$$\begin{aligned} f_{0,0} &= \log \frac{4i\sqrt{\pi}\sqrt{\rho}}{(1-\rho)^{3/2}}, & g_{0,0} &= \log \left(-\frac{4i\sqrt{\pi}\sqrt{\rho}}{(1-\rho)^{3/2}} \right), & h_{0,0} &= \log \left(\frac{2\sqrt{\pi}\sqrt{\rho}}{(1-\rho)^{3/2}} \right), \\ f_1(v) &= \frac{1}{\sqrt{v}} + \frac{i}{\sqrt{\rho-v}} - \frac{2\sqrt{i}\sqrt{\pi}\sqrt{\rho}}{\sqrt{1-\rho}(\rho + (1-\rho)\Phi(v - \rho/2))}, \\ g_1(v) &= \frac{1}{\sqrt{v}} - \frac{i}{\sqrt{\rho-v}} - \frac{2\sqrt{-i}\sqrt{\pi}\sqrt{\rho}}{\sqrt{1-\rho}(\rho + (1-\rho)\Phi(\rho/2 - v))}, \\ f_2(v) &= -\frac{1}{2(\rho-v)} + \frac{\sqrt{\rho-v}}{2\sqrt{\rho}v} - \frac{2i\pi\rho}{(1-\rho)(\rho + (1-\rho)\Phi(v - \rho/2))^2} + \frac{4i\pi(1+\rho)}{3(1-\rho)(\rho + (1-\rho)\Phi(v - \rho/2))}, \\ g_2(v) &= -\frac{1}{2(\rho-v)} + \frac{\sqrt{\rho-v}}{2\sqrt{\rho}v} + \frac{2i\pi\rho}{(1-\rho)(\rho + (1-\rho)\Phi(\rho/2 - v))^2} - \frac{4i\pi(1+\rho)}{3(1-\rho)(\rho + (1-\rho)\Phi(\rho/2 - v))}. \end{aligned} \quad (127)$$

When the two arguments of f are equal, one finds the limits

$$\begin{aligned} f(v, v) &= \log \left(\frac{8\pi\sqrt{v}\sqrt{\rho-v}\Phi(v - \rho/2)(1 - \Phi(v - \rho/2))}{(\sqrt{-iv} + \sqrt{i(\rho-v)})(\rho + (1-\rho)\Phi(v - \rho/2))} \right), \\ f^{(1,0)}(v, v) &= -\frac{1}{2\rho} + \frac{i\pi}{1-\rho} + \frac{1}{4v} - \frac{1}{4(\rho-v)} + \frac{i\sqrt{\rho-v}}{2\rho\sqrt{v}} - \frac{i\pi\rho}{(1-\rho)(\rho + (1-\rho)\Phi(v - \rho/2))^2}, \\ f^{(0,1)}(v, v) &= \frac{1}{2\rho} + \frac{i\pi}{1-\rho} + \frac{1}{4v} - \frac{1}{4(\rho-v)} + \frac{i\sqrt{v}}{2\rho\sqrt{\rho-v}} - \frac{i\pi\rho}{(1-\rho)(\rho + (1-\rho)\Phi(v - \rho/2))^2}. \end{aligned} \quad (128)$$

One can use (87) for the asymptotic expansion. After some simplifications, which involve in particular the calculation of several simple and double integrals explained in appendix B, and using the explicit values (141) and (155) for $\zeta(0, z)$, $\zeta(-1, z)$ and $\tilde{\zeta}_0(0, 0, z, z')$, one finds

$$\begin{aligned} S_{\text{reg}} &\simeq L^2 \left(\frac{9\rho^2}{8} + \frac{\rho^2 \log \rho}{4} + \frac{\rho(1-\rho) \log(1-\rho)}{2} \right) \\ &+ L \left(-\frac{\rho \log \rho}{4} - \frac{(1-\rho) \log(1-\rho)}{2} + \frac{\rho}{2} - \frac{\rho \log(8\pi)}{2} + \frac{\pi \rho c}{2} \right) \\ &+ \sqrt{L} \left(2\sqrt{\rho} - \frac{\sqrt{2\pi}\sqrt{\rho}}{\sqrt{1-\rho}} \right) \left((1+i)\zeta(-\tfrac{1}{2}, \tfrac{1}{2} + ic) + (1-i)\zeta(-\tfrac{1}{2}, \tfrac{1}{2} - ic) \right) + \left(-\frac{1}{8} + \frac{\log 2}{12} - \frac{\pi c}{2} - c^2 \right). \end{aligned} \quad (129)$$

The singular part S_{sing} of the sum in (124) is equal to

$$\begin{aligned}
& \sum_{j=1}^N \sum_{j'=j+1}^N \log \frac{\left(\frac{j'-j}{L}\right) \left(\sqrt{-i \frac{j-\frac{1}{2}+ic}{L}} + \sqrt{i \frac{j'-\frac{1}{2}+ic}{L}}\right)}{\left(\sqrt{\frac{j-\frac{1}{2}+ic}{L}} + \sqrt{\frac{j'-\frac{1}{2}+ic}{L}}\right) \left(\sqrt{\rho - \frac{j-\frac{1}{2}+ic}{L}} + \sqrt{\rho - \frac{j'-\frac{1}{2}+ic}{L}}\right)} \\
&= N \log 2 - \frac{N(N-1)}{4} \log L + \log(G(N+1)) + \frac{1}{4} \log \frac{\Gamma(N + \frac{1}{2} + ic) \Gamma(N + \frac{1}{2} - ic)}{\Gamma(\frac{1}{2} + ic) \Gamma(\frac{1}{2} - ic)} \\
&+ \sum_{j=1}^N \sum_{j'=1}^N \log \left(\sqrt{-i(j - \frac{1}{2} + ic)} + \sqrt{i(j' - \frac{1}{2} - ic)} \right) \\
&- \sum_{j=1}^N \sum_{j'=1}^N 1_{\{j+j' > N\}} \log \left(\sqrt{-i(j - \frac{1}{2} + ic)} + \sqrt{i(j' - \frac{1}{2} - ic)} \right) \\
&- \frac{1}{2} \sum_{j=1}^N \sum_{j'=1}^N \log \left(\sqrt{j - \frac{1}{2} + ic} + \sqrt{j' - \frac{1}{2} + ic} \right) - \frac{1}{2} \sum_{j=1}^N \sum_{j'=1}^N \log \left(\sqrt{j - \frac{1}{2} - ic} + \sqrt{j' - \frac{1}{2} - ic} \right),
\end{aligned} \tag{130}$$

where G is Barnes function $G(N+1) = \prod_{j=1}^{N-1} j!$, whose large N expansion is given by (89). The expansions of the remaining sums are obtained from (95), (90) and

$$\begin{aligned}
\sum_{j=1}^N \sum_{j'=1}^N 1_{\{j+j' > N\}} \log \left(\sqrt{-i(j - \frac{1}{2} + ic)} + \sqrt{i(j' - \frac{1}{2} - ic)} \right) &\simeq \frac{N^2 \log N}{4} + \left(\log 2 - \frac{5}{8} \right) N^2 \\
&+ \frac{N \log N}{4} - \frac{(4-\pi)c}{2} N - \left(\frac{1 + \log 2}{24} - \frac{\pi c}{4} \right),
\end{aligned} \tag{131}$$

which can be derived by cutting the triangle into two triangles and a rectangle and using (77) and (79).

The expansions (95) and (90) contribute the constants $\kappa_1(-1)$ and κ_0 as $\kappa_1(-1) - \kappa(0) \approx 0.18195074673927841$. These constants can be evaluated numerically with very large precision using BST algorithm as described in section 3.9. From numerical computations, we conjecture the identity

$$\kappa_1(-1) - \kappa(0) = \frac{1}{12} - \zeta'(-1) - \frac{\log 2}{8} - \int_{-\infty}^0 du \left(e^{i\pi/4} \zeta\left(\frac{1}{2}, \frac{1}{2} + iu\right) - e^{-i\pi/4} \zeta\left(\frac{1}{2}, \frac{1}{2} - iu\right) \right)^2, \tag{132}$$

that was checked within 100 significant digits. Our derivation of the asymptotics (36) of the norm $\mathcal{N}_r(\gamma)$ does not rely heavily on this numerical conjecture since the value of $\kappa_1(-1) - \kappa(0)$ can in fact also be inferred from the stationary value $\mathcal{N}_0(0) = 1$.

Gathering the various contributions to the singular term, one finds

$$\begin{aligned}
S_{\text{sing}} &\simeq \left(\frac{\log \rho}{4} - \frac{9}{8} \right) \rho^2 L^2 + \frac{\rho L \log L}{2} + \left(\frac{\log \rho}{4} - \frac{1}{2} + \frac{\log(8\pi)}{2} + \frac{\pi c}{2} \right) \rho L \\
&- 2\sqrt{\rho}\sqrt{L} \left((1+i)\zeta\left(-\frac{1}{2}, \frac{1}{2} + ic\right) + (1-i)\zeta\left(-\frac{1}{2}, \frac{1}{2} - ic\right) \right) \\
&+ \frac{1}{8} - \frac{\log 2}{12} + \frac{\log(2\pi)}{4} - \frac{\pi c}{4} + c^2 - \frac{\log \Gamma(\frac{1}{2} + ic)}{4} - \frac{\log \Gamma(\frac{1}{2} - ic)}{4} \\
&- \frac{1}{4} \int_{-\infty}^c du \left(e^{i\pi/4} \zeta\left(\frac{1}{2}, \frac{1}{2} + iu\right) - e^{-i\pi/4} \zeta\left(\frac{1}{2}, \frac{1}{2} - iu\right) \right)^2.
\end{aligned} \tag{133}$$

Putting the regular and the singular terms together, taking the exponential, and using Euler's reflection formula $\Gamma(z)\Gamma(1-z) = \pi/\sin(\pi z)$ to eliminate the Γ functions, we finally obtain

$$\begin{aligned} \prod_{j=1}^N \prod_{j'=j+1}^N (y_j^0 - y_{j'}^0) &\simeq i^{\frac{N(N-1)}{2}} e^{\frac{\rho b L^2}{2}} e^{\frac{\rho L \log L}{2}} e^{-\frac{(1-\rho) \log(1-\rho) L}{2}} \\ &\times \exp \left(-\frac{2\sqrt{\pi}\sqrt{\rho}}{\sqrt{1-\rho}} (e^{i\pi/4} \zeta(-\frac{1}{2}, \frac{1}{2} + ic) + e^{-i\pi/4} \zeta(-\frac{1}{2}, \frac{1}{2} - ic)) \sqrt{L} \right) \\ &\times e^{-\pi c} (1 + e^{2\pi c})^{1/4} \exp \left[-\frac{1}{4} \int_{-\infty}^c du \left(e^{i\pi/4} \zeta(\frac{1}{2}, \frac{1}{2} + iu) - e^{-i\pi/4} \zeta(\frac{1}{2}, \frac{1}{2} - iu) \right)^2 \right], \end{aligned} \quad (134)$$

where $(1 + e^{2\pi c})^{1/4}$ has to be interpreted once again as the analytic continuation in c from the real axis as explained after (112).

5.6 Asymptotics of $\Xi_1^{-1} \Xi_2^{-1} \Xi_3^2 \Xi_4^2$

We observe that many simplifications occur when multiplying Ξ_3 and Ξ_4 if we replace the ζ function by the eigenstate-dependent function φ_r defined in (28). One has

$$\begin{aligned} \prod_{j=1}^N \prod_{k=j+1}^N (y_j - y_k) &\simeq i^{\frac{N(N-1)}{2}} e^{\frac{\rho b L^2}{2}} e^{\frac{\rho L \log L}{2}} e^{-\frac{(1-\rho) \log(1-\rho) L}{2}} e^{-\frac{\sqrt{\rho} \varphi_r(2\pi c) \sqrt{L}}{\sqrt{1-\rho}}} e^{-\frac{2i\pi(1-2\rho)p_r}{3(1-\rho)}} \\ &\frac{(\pi/2)^{m_r^2}}{(2\pi)^{m_r}} e^{i\pi(\sum_{a \in A_0^+} a - \sum_{a \in A_0^-} a)} \omega(A_0^+) \omega(A_0^-) \omega(A^+) \omega(A^-) \omega(A_0^+, A_0^-) \omega(A^+, A^-) \\ &\frac{(1 + e^{2\pi c})^{1/4}}{e^{\pi c}} \frac{(\prod_{a \in A^+} (c - ia)^{1/4}) (\prod_{a \in A^-} (c + ia)^{1/4})}{(\prod_{a \in A_0^+} (c + ia)^{1/4}) (\prod_{a \in A_0^-} (c - ia)^{1/4})} \exp \left(\lim_{\Lambda \rightarrow \infty} -m_r^2 \log \Lambda + \int_{-\Lambda}^{2\pi c} du \frac{(\varphi'_r(u))^2}{2} \right), \end{aligned} \quad (135)$$

where the combinatorial factors ω are defined in (37). The limit $\Lambda \rightarrow \infty$ is needed to define the integral, which is divergent when $u = -\infty$ (except for the stationary state $m_r^+ = m_r^- = 0$) since $\varphi'_r(u) \sim u^{-1/2}$ when $u \rightarrow -\infty$. Even more simplifications occur after dividing $\Xi_3^2 \Xi_4^2$ by $\Xi_1 \Xi_2$. One finally finds

$$\begin{aligned} \frac{\prod_{j=1}^N \prod_{k=j+1}^N (y_j - y_k)^2}{\left(\frac{1}{N} \sum_{j=1}^N \frac{y_j}{\rho + (1-\rho)y_j} \right) \left(\prod_{j=1}^N \left(1 + \frac{1-\rho}{\rho} y_j \right) \right)} &\simeq e^{\rho b L^2} e^{\rho L \log L} e^{-(1-\rho) \log(1-\rho) L} \exp \left(-\frac{2\sqrt{\rho} \varphi_r(2\pi c)}{\sqrt{1-\rho}} \sqrt{L} \right) \\ &\times \frac{(-1)^{\frac{N(N-1)}{2} + m_r}}{(\pi^2/4)^{-m_r^2} (4\pi^2)^{m_r}} \exp \left(-\frac{4i\pi(1-2\rho)p_r}{3(1-\rho)} \right) \omega(A_0^+)^2 \omega(A_0^-)^2 \omega(A^+)^2 \omega(A^-)^2 \omega(A_0^+, A_0^-)^2 \omega(A^+, A^-)^2 \\ &\times \frac{\sqrt{\rho(1-\rho)} \sqrt{L}}{e^{2\pi c} \varphi'_r(2\pi c)} \exp \left(\lim_{\Lambda \rightarrow \infty} -2m_r^2 \log \Lambda + \int_{-\Lambda}^{2\pi c} du (\varphi'_r(u))^2 \right). \end{aligned} \quad (136)$$

We observe that several factors depending on c have cancelled: the only dependency on c left are $\varphi_r(2\pi c)$, $e^{2\pi c} \varphi'_r(2\pi c)$, and the upper limit of the integral.

6 Conclusions

It was shown in [29] that the first eigenvalues of TASEP are naturally expressed in terms of a function η constructed from the elementary excitations characterizing the eigenstate. We extend that result here by showing that $\varphi = \eta'$ can be identified as a scalar field: indeed, we observe that the normalization of the corresponding Bethe eigenstates can be expressed in terms of the exponential of the free action of φ . This might hint at a field theoretic description of current fluctuations on the relaxation scale $T \sim L^{3/2}$.

The field φ is equal to a sum of square roots corresponding to elementary excitations over a Fermi sea, plus Hurwitz zeta functions corresponding to a kind of renormalization of the contribution of the

Fermi sea. In our calculations, the contributions leading to these two parts of the field need to be treated separately. It is only at the end of the calculation that everything combines perfectly at several places to give exactly the same field φ everywhere. It would be very nice to find a simpler derivation that makes it clearer why the field φ should appear in the end, and to explain all the other unexpected cancellations that happen between the asymptotic expansions of seemingly very different factors.

A Hurwitz zeta function and double Hurwitz zeta function

In this appendix, we summarize some properties of Hurwitz zeta function and double Hurwitz zeta function.

A.1 Hurwitz zeta function

A.1.1 Definitions

Hurwitz zeta function is defined for $\operatorname{Re} s > 1$ by

$$\zeta(s, z) = \sum_{j=0}^{\infty} (j+z)^{-s} . \quad (137)$$

For $z \notin \mathbb{R}^-$, it can be analytically continued to a meromorphic function of s with a simple pole at $s = 1$ with residue equal to 1, independent of z . It is convenient for using the Euler-Maclaurin formula to define a modification $\tilde{\zeta}$ of ζ such that $\tilde{\zeta}(s, z) = \zeta(s, z)$ when $s \neq 1$ and

$$\tilde{\zeta}(1, z) = \lim_{s \rightarrow 1} \zeta(s, z) - \frac{1}{s-1} = -\frac{\Gamma'(z)}{\Gamma(z)} . \quad (138)$$

The modified function $\tilde{\zeta}$ is not continuous at $s = 1$. It obeys however the property

$$\lim_{s \rightarrow 1} (\zeta(s, z) - \zeta(s, z')) = \tilde{\zeta}(s, z) - \tilde{\zeta}(s, z') . \quad (139)$$

A.1.2 Derivative

Hurwitz zeta function verifies

$$\partial_z \zeta(s, z) = -s \zeta(s+1, z) . \quad (140)$$

A.1.3 Bernoulli polynomials

Hurwitz zeta function is related to Bernoulli polynomials. For $r \in \mathbb{N}^*$, one has

$$B_r(z) = -r \zeta(1-r, z) , \quad (141)$$

which is a polynomial in z of degree r . The Bernoulli polynomials form an Appell sequence, *i.e.* $B'_r(x) = r B_{r-1}(x)$, or equivalently

$$B_r(x+y) = \sum_{m=0}^r \binom{r}{m} B_m(x) y^{r-m} . \quad (142)$$

They also verify the symmetry relation

$$B_r(x+1) = (-1)^r B_r(-x) . \quad (143)$$

A.1.4 Asymptotic expansions

When its second argument becomes large, Hurwitz zeta has the asymptotic expansion

$$\zeta(s, M+x) \simeq -\frac{1}{1-s} \sum_{\ell=0}^{\infty} \binom{1-s}{\ell} \frac{B_{\ell}(x)}{M^{\ell+s-1}}, \quad (144)$$

while for $\tilde{\zeta}$, Stirling's formula leads to

$$\tilde{\zeta}(1, M+x) \simeq -\log M + \sum_{\ell=1}^{\infty} \frac{(-1)^{\ell}}{\ell} \frac{B_{\ell}(x)}{M^{\ell}}. \quad (145)$$

A.2 Double Hurwitz zeta function

A two-dimensional generalization, the double Hurwitz zeta function, can be defined as

$$\zeta(s, s'; z, z') = \sum_{j=0}^{\infty} \sum_{j'=j+1}^{\infty} (j+z)^{-s} (j'+z')^{-s'}. \quad (146)$$

The sum converges for z and z' outside \mathbb{R}^- when $\operatorname{Re} s' > 1$ and $\operatorname{Re}(s+s') > 2$. Unlike the usual Hurwitz zeta function, there does not seem to be a standard accepted notation here, partly due to the fact that several natural two dimensional generalizations can be considered.

A.2.1 Analytic continuation

The analytic continuation to arbitrary s, s' can be made [41] using the Mellin-Barnes integral formula, which can be stated for $-\operatorname{Re} s < p < 0$, $\lambda \notin \mathbb{R}^-$ as

$$(1+\lambda)^{-s} = \int_{p-i\infty}^{p+i\infty} \frac{dw}{2i\pi} \frac{\Gamma(s+w)\Gamma(-w)}{\Gamma(s)} \lambda^w, \quad (147)$$

and which follows from closing the contour of integration on the right and calculating the residues on the positive real axis when $|\lambda| < 1$. Rewriting (146) as

$$\zeta(s, s'; z, z') = \sum_{j=0}^{\infty} \sum_{j'=0}^{\infty} (j+z)^{-s} (j'+z'-z+1)^{-s'} \left(1 + \frac{j+z}{j'+z'-z+1}\right)^{-s'}, \quad (148)$$

applying (147) to the factor $(1 + \frac{j+z}{j'+z'-z+1})^{-s'}$, shifting w by $-s'$, and using (47) to compute the sums over j and j' (provided $1 < \operatorname{Re} w < \operatorname{Re}(s+s'-1)$) leads to

$$\zeta(s, s'; z, z') = \int_{p-i\infty}^{p+i\infty} \frac{dw}{2i\pi} \frac{\Gamma(w)\Gamma(s'-w)\zeta(s+s'-w, z)\zeta(w, z'-z+1)}{\Gamma(s')} \quad (149)$$

with $1 < p < \operatorname{Re} s'$. The contour of integration can be shifted to $0 < p < 1$ by taking the residue coming from the simple pole with residue 1 of $\zeta(w, z'-z+1)$ at $w=1$. Shifting again the contour of integration to the left we pick the residues at $w=-k$, $k \in \mathbb{N}$ coming from $\Gamma(w)$ (residue $(-1)^k/k!$ at $w=-k$). Shifting k by 1, one finally finds in terms of Bernoulli polynomials (141)

$$\begin{aligned} \zeta(s, s'; z, z') &= \frac{1}{s'-1} \sum_{\ell=0}^{m+1} \binom{1-s'}{\ell} \zeta(s+s'+\ell-1, z) B_{\ell}(z'-z+1) \\ &\quad + \int_{-m-\frac{1}{2}-i\infty}^{-m-\frac{1}{2}+i\infty} \frac{dw}{2i\pi} \frac{\Gamma(w)\Gamma(s'-w)\zeta(s+s'-w, z)\zeta(w, z'-z+1)}{\Gamma(s')}. \end{aligned} \quad (150)$$

The remaining integral is analytic in the domain $\{(s, s'), \operatorname{Re} s' > -m - \frac{1}{2}, \operatorname{Re}(s + s') > -m + \frac{1}{2}\}$. It implies that $\zeta(s, s'; z, z')$ is a meromorphic function of s, s' with (possible) poles at $s' = 1$ and $s + s' = 2 - n$, $n \in \mathbb{N}$. When approaching the pole at $s + s' = 2 - n$, one has (when $s, s' \notin 1 - \mathbb{N}$)

$$\begin{aligned} \zeta(s + \alpha\varepsilon, s' + (1 - \alpha)\varepsilon; z, z') &\underset{\varepsilon \rightarrow 0}{\sim} \frac{1}{s' - 1} \binom{1 - s'}{n} B_n(z' - z + 1) \\ &\times \left(\frac{1}{\varepsilon} - \frac{\Gamma'(z)}{\Gamma(z)} - (1 - \alpha) \frac{\Gamma'(1 - s')}{\Gamma(1 - s')} + (1 - \alpha) \frac{\Gamma'(s)}{\Gamma(s)} \right). \end{aligned} \quad (151)$$

The arbitrary parameter α characterizes the direction in which (s, s') approaches the line $s + s' = 2 - n$. We define a modified version $\tilde{\zeta}_\alpha$ of double Hurwitz zeta, equal to ζ when $2 - s - s' \notin \mathbb{N}$, and to

$$\tilde{\zeta}_\alpha(s, s'; z, z') = \lim_{\varepsilon \rightarrow 0} \left(\zeta(s + \alpha\varepsilon, s' + (1 - \alpha)\varepsilon; z, z') - \frac{1}{s' - 1} \binom{1 - s'}{n} \frac{B_n(z' - z + 1)}{\varepsilon} \right) \quad (152)$$

when $s + s' = 2 - n$, $n \in \mathbb{N}$. More explicitly

$$\begin{aligned} \tilde{\zeta}_\alpha(s, s'; z, z') &= \frac{1}{s' - 1} \sum_{\substack{\ell=0 \\ (\ell \neq n)}}^{m+1} \binom{1 - s'}{\ell} \zeta(\ell - n + 1, z) B_\ell(z' - z + 1) \\ &\quad - \frac{1}{s' - 1} \binom{1 - s'}{n} B_n(z' - z + 1) \left(\frac{\Gamma'(z)}{\Gamma(z)} + (1 - \alpha) \left(\frac{\Gamma'(1 - s')}{\Gamma(1 - s')} - \frac{\Gamma'(s)}{\Gamma(s)} \right) \right) \\ &\quad + \int_{-m - \frac{1}{2} - i\infty}^{-m - \frac{1}{2} + i\infty} \frac{dw}{2i\pi} \frac{\Gamma(w) \Gamma(s' - w) \zeta(s + s' - w, z) \zeta(w, z' - z + 1)}{\Gamma(s')}, \end{aligned} \quad (153)$$

with $m \geq n - 1$, $m \geq -\operatorname{Re} s' - \frac{1}{2}$ and when $s, s' \notin 1 - \mathbb{N}$. For $\alpha = 1$, this corresponds to replacing the simple Hurwitz zeta function in the summation in (150) by its modified value (138).

A.2.2 Double Bernoulli polynomials

The modified double zeta functions can be extended to $s \in -\mathbb{N}$, $s' \in -\mathbb{N}$. There, the integral vanishes because of the $\Gamma(s')$ in the denominator and $\tilde{\zeta}_\alpha(s, s'; z, z')$ become polynomials in z, z' . Using (141), we obtain ($s + s' = 2 - n$)

$$\tilde{\zeta}_\alpha(s, s'; z, z') = \frac{1}{s' - 1} \sum_{\ell=0}^{n-1} \binom{1 - s'}{\ell} \frac{B_\ell(z' - z + 1) B_{n-\ell}(z)}{\ell - n} + B_n(z' - z + 1) \frac{(-1)^s (1 - \alpha)}{(1 - s)(1 - s') \binom{2-s-s'}{1-s}}. \quad (154)$$

In particular, at $s = s' = 0$, one has

$$\tilde{\zeta}_\alpha(0, 0; z, z') = -\frac{1 + \alpha}{12} + \frac{\alpha(z - z')}{2} - \frac{\alpha z^2}{2} + \frac{(1 - \alpha)(z')^2}{2} + \alpha z z'. \quad (155)$$

Unlike the one-dimensional case, there is no unique natural way to define double Bernoulli numbers and polynomials because of the arbitrary parameter α .

A.2.3 Asymptotic expansion

The expressions (150) gives the large M asymptotic expansion of $\zeta(s, s'; z + M, z' + M)$ using the one (144) for simple Hurwitz zeta since $\zeta(s + s' - w, M + z) \sim M^{-m + \frac{1}{2} - s - s'}$ in the integral can be made arbitrarily small by taking m large enough. This gives the asymptotic expansion

$$\zeta(s, s'; M + z, M + z') \simeq \sum_{\ell=0}^{\infty} \sum_{m=0}^{\infty} \frac{\binom{1 - s'}{\ell}}{1 - s'} \frac{\binom{2-s-s'-\ell}{m}}{2 - s - s' - \ell} \frac{B_m(z) B_\ell(z' - z + 1)}{M^{\ell+m+s+s'-2}}, \quad (156)$$

valid when $s' \neq 1$ and $s + s' \notin 2 - \mathbb{N}$, and similarly when $s + s' \in 2 - \mathbb{N}$ from (153) and (154).

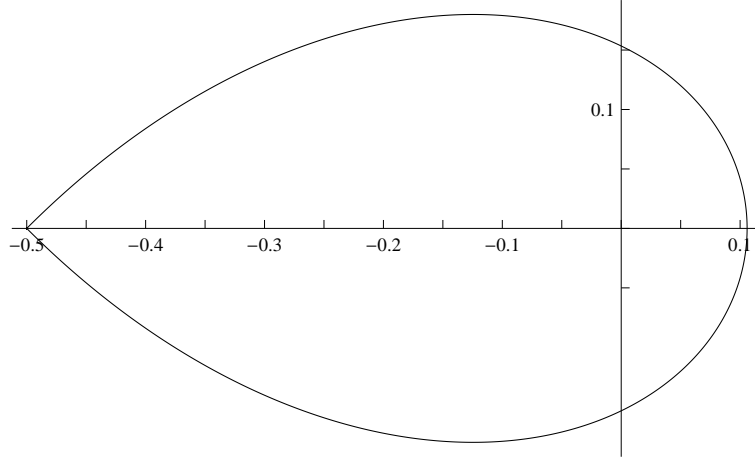


Fig. 2 Contour on which the Bethe roots y_j accumulate in the complex plane for the first eigenstates in the thermodynamic limit with density of particles $\rho = 1/3$. It crosses the negative real axis at $-\rho/(1-\rho)$. For any value of ρ , the contour encloses 0 but not 1.

B Calculation of various integrals

The various asymptotic expansions obtained in this paper using the Euler-Maclaurin formula involve integrals. Most of them have an integrand that depends on the variable of integration $u \in [0, \rho]$ only through $\Phi(u - \frac{\rho}{2})$, with Φ defined in (100). Such integrals can be computed by making the change of variables $z = \Phi(u - \frac{\rho}{2})$. From (104), the Jacobian is given by

$$du = -\frac{dz}{2i\pi} \left(\frac{\rho}{z} + \frac{1}{1-z} \right). \quad (157)$$

The variable z lives on the clockwise contour $\overline{\mathcal{C}_0}$, which starts and ends at $z = -\rho/(1-\rho)$ for $u = 0$ and $u = \rho$. The contour encloses 0 but not 1, see figure 2. Hence, for the simplest integrands that do not involve branch cuts, the calculation of the integral reduces to a simple residue calculation.

In the rest of this appendix, we treat some slightly more complicated integrals on two dimensional domains, with integrands having branch cuts that cross the contour of integration. We use the notation $\mathcal{C}_0 = \{\Phi(\frac{\rho}{2} - u), 0 \leq u \leq \rho\}$ for the counter clockwise contour corresponding to $\overline{\mathcal{C}_0}$.

B.1 A double integral

We consider the double integral

$$I_1 = \int_0^\rho du \int_u^\rho dv \log \left(-i(\Phi(u - \frac{\rho}{2}) - \Phi(v - \frac{\rho}{2})) \right). \quad (158)$$

Making the changes of variables $z = \Phi(u - \frac{\rho}{2})$ and $w = \Phi(v - \frac{\rho}{2})$ leads to

$$I_1 = \int_{\overline{\mathcal{C}_0}} \frac{dz}{2i\pi} \left(\frac{\rho}{z} + \frac{1}{1-z} \right) \int_z^{-\frac{\rho}{1-\rho}} \frac{dw}{2i\pi} \left(\frac{\rho}{w} + \frac{1}{1-w} \right) \log(-i(z-w)). \quad (159)$$

The inner integral can be computed in terms of the dilogarithm function Li_2 . Indeed, using $\text{Li}_2'(w) = -w^{-1} \log(1-w)$, we observe that the function F_z defined by

$$F_z(w) = \rho \left(\text{Li}_2\left(\frac{z}{w}\right) + \frac{(\log w)^2}{2} + \frac{i\pi}{2} \log w \right) - \left(\text{Li}_2\left(\frac{1-z}{1-w}\right) + \frac{(\log(1-w))^2}{2} - \frac{i\pi}{2} \log(1-w) \right) \quad (160)$$

verifies

$$F'_z(w) = \left(\frac{\rho}{w} + \frac{1}{1-w} \right) \log(-i(z-w)) . \quad (161)$$

The contour of integration for w in (159) does not cross the branch cuts coming from the dilogarithm. The integration over z can be done by taking the residue at 0 for all the terms such that the contour does not cross a branch cut. Using $\text{Li}_2(0) = 0$, one finds

$$I_1 = \frac{\rho}{2} \int_{\mathcal{C}_0} \frac{dz}{(2i\pi)^2} \left(\frac{\rho}{z} + \frac{1}{1-z} \right) ((\log z)^2 + i\pi \log z) \quad (162)$$

$$+ \frac{\rho}{2i\pi} \left(\frac{i\pi b_0}{2} - \frac{(1-\rho)\pi^2}{6} + \frac{(1-\rho)(\log(1-\rho))^2}{2} - \frac{\rho(\log \rho)^2}{2} + \rho \log \rho \log(1-\rho) + \text{Li}_2(1-\rho) \right) .$$

The last integral can be computed using the fact that the contour \mathcal{C}_0 intersects the negative real axis at $z = -\rho/(1-\rho)$ and the identities $(\log z)/z = \partial_z(\log z)^2/2$, $(\log z)^2/z = \partial_z(\log z)^3/3$, $(\log z)/(z-1) = \partial_z(\text{Li}_2(z) + \log z \log(1-z))$ and $(\log z)^2/(z-1) = \partial_z(-2\text{Li}_3(z) + 2\log z \text{Li}_2(z) + (\log z)^2 \log(1-z))$. After some simplifications, one finds

$$I_1 = \frac{\rho b_0}{2} , \quad (163)$$

with b_0 defined in (31).

B.2 Another double integral

We consider the double integral

$$I_2 = \int_0^\rho du \int_u^\rho dv \log \frac{(\sqrt{u} + \sqrt{v})(\sqrt{\rho-u} + \sqrt{\rho-v})}{(v-u)(\sqrt{-iu} + \sqrt{i(\rho-v)})} . \quad (164)$$

It can be rewritten as

$$I_2 = \int_0^\rho du \int_u^\rho dv \log(\sqrt{u} + \sqrt{v}) - \int_0^\rho du \int_u^\rho dv \log(v-u) - \int_0^\rho du \int_u^\rho dv \log(\sqrt{-iu} + \sqrt{i(\rho-v)}) . \quad (165)$$

Using $\log(\sqrt{u} + \sqrt{v}) = \partial_v((v-u)\log(\sqrt{u} + \sqrt{v}) + \sqrt{uv} - v/2)$ and $\log(\sqrt{-iu} + \sqrt{i(\rho-v)}) = \partial_v(-(u + \rho - v)\log(\sqrt{-iu} + \sqrt{i(\rho-v)}) + i\sqrt{u}\sqrt{\rho-v} - v/2)$, one finds

$$I_2 = \frac{9\rho^2}{8} - \frac{\rho^2 \log \rho}{4} . \quad (166)$$

References

1. T. Halpin-Healy and Y.-C. Zhang. Kinetic roughening phenomena, stochastic growth, directed polymers and all that. Aspects of multidisciplinary statistical mechanics. *Phys. Rep.*, 254:215–414, 1995.
2. T. Sasamoto and H. Spohn. The 1+1-dimensional Kardar-Parisi-Zhang equation and its universality class. *J. Stat. Mech.*, 2010:P11013.
3. T. Kriecherbauer and J. Krug. A pedestrian's view on interacting particle systems, KPZ universality and random matrices. *J. Phys. A: Math. Theor.*, 43:403001, 2010.
4. I. Corwin. The Kardar-Parisi-Zhang equation and universality class. *Random Matrices: Theory and Applications*, 1:1130001, 2011.
5. B. Derrida. An exactly soluble non-equilibrium system: the asymmetric simple exclusion process. *Phys. Rep.*, 301:65–83, 1998.
6. O. Golinelli and K. Mallick. The asymmetric simple exclusion process: an integrable model for non-equilibrium statistical mechanics. *J. Phys. A: Math. Gen.*, 39:12679–12705, 2006.
7. K. Johansson. Shape fluctuations and random matrices. *Commun. Math. Phys.*, 209:437–476, 2000.
8. M. Prähofer and H. Spohn. Universal distributions for growth processes in 1+1 dimensions and random matrices. *Phys. Rev. Lett.*, 84:4882, 2000.

9. A. Borodin, I. Corwin, and P. Ferrari. Free energy fluctuations for directed polymers in random media in 1+1 dimension. *Comm. Pure Appl. Math.*, 67:1129–1214, 2014.
10. M. Kardar, G. Parisi, and Y.-C. Zhang. Dynamic scaling of growing interfaces. *Phys. Rev. Lett.*, 56:889–892, 1986.
11. P. Calabrese, P. Le Doussal, and A. Rosso. Free-energy distribution of the directed polymer at high temperature. *Europhys. Lett.*, 90:20002, 2010.
12. V. Dotsenko. Bethe ansatz derivation of the Tracy-Widom distribution for one-dimensional directed polymers. *Europhys. Lett.*, 90:20003, 2010.
13. B. Derrida and J.L. Lebowitz. Exact large deviation function in the asymmetric exclusion process. *Phys. Rev. Lett.*, 80:209–213, 1998.
14. E. Brunet and B. Derrida. Probability distribution of the free energy of a directed polymer in a random medium. *Phys. Rev. E*, 61:6789–6801, 2000.
15. M. Gorissen, A. Lazarescu, K. Mallick, and C. Vanderzande. Exact current statistics of the asymmetric simple exclusion process with open boundaries. *Phys. Rev. Lett.*, 109:170601, 2012.
16. D.S. Lee and D. Kim. Universal fluctuation of the average height in the early-time regime of one-dimensional Kardar-Parisi-Zhang-type growth. *J. Stat. Mech.*, 2006:P08014.
17. J.G. Brankov, V.V. Papoyan, V.S. Poghosyan, and V.B. Priezzhev. The totally asymmetric exclusion process on a ring: Exact relaxation dynamics and associated model of clustering transition. *Physica A*, 368:471480, 2006.
18. S. Gupta, S.N. Majumdar, C. Godrèche, and M. Barma. Tagged particle correlations in the asymmetric simple exclusion process: Finite-size effects. *Phys. Rev. E*, 76:021112, 2007.
19. N.M. Bogoliubov. Determinantal representation of the time-dependent stationary correlation function for the totally asymmetric simple exclusion model. *SIGMA*, 5:052, 2009.
20. A. Proeme, R.A. Blythe, and M.R. Evans. Dynamical transition in the open-boundary totally asymmetric exclusion process. *J. Phys. A: Math. Theor.*, 44:035003, 2011.
21. K. Motegi, K. Sakai, and J. Sato. Exact relaxation dynamics in the totally asymmetric simple exclusion process. *Phys. Rev. E*, 85:042105, 2012.
22. K. Motegi, K. Sakai, and J. Sato. Long time asymptotics of the totally asymmetric simple exclusion process. *J. Phys. A: Math. Theor.*, 45:465004, 2012.
23. S. Prolhac. Current fluctuations for totally asymmetric exclusion on the relaxation scale. *J. Phys. A: Math. Theor.*, 48:06FT02, 2015.
24. H. Spohn. Stochastic integrability and the KPZ equation. *IAMP news bulletin*, pages 5–9, April 2012.
25. L.-H. Gwa and H. Spohn. Six-vertex model, roughened surfaces, and an asymmetric spin Hamiltonian. *Phys. Rev. Lett.*, 68:725–728, 1992.
26. O. Golinelli and K. Mallick. Bethe ansatz calculation of the spectral gap of the asymmetric exclusion process. *J. Phys. A: Math. Gen.*, 37:3321–3331, 2004.
27. O. Golinelli and K. Mallick. Spectral gap of the totally asymmetric exclusion process at arbitrary filling. *J. Phys. A: Math. Gen.*, 38:1419–1425, 2005.
28. S. Prolhac. Spectrum of the totally asymmetric simple exclusion process on a periodic lattice - bulk eigenvalues. *J. Phys. A: Math. Theor.*, 46:415001, 2013.
29. S. Prolhac. Spectrum of the totally asymmetric simple exclusion process on a periodic lattice - first excited states. *J. Phys. A: Math. Theor.*, 47:375001, 2014.
30. C.A. Tracy and H. Widom. Integral formulas for the asymmetric simple exclusion process. *Commun. Math. Phys.*, 279:815–844, 2008.
31. R.I. Nepomechie and C. Wang. Twisting singular solutions of Bethe’s equations. *J. Phys. A: Math. Theor.*, 47:505004, 2014.
32. K. Motegi and K. Sakai. Vertex models, TASEP and Grothendieck polynomials. *J. Phys. A: Math. Theor.*, 46:355201, 2013.
33. N.A. Slavnov. Calculation of scalar products of wave functions and form factors in the framework of the algebraic Bethe ansatz. *Theor. Math. Phys.*, 79:502–508, 1989.
34. M. Gaudin, B.M. McCoy, and T.T. Wu. Normalization sum for the Bethe’s hypothesis wave functions of the Heisenberg-Ising chain. *Phys. Rev. D*, 23:417–419, 1981.
35. V.E. Korepin. Calculation of norms of Bethe wave functions. *Commun. Math. Phys.*, 86:391–418, 1982.
36. G.M. Schütz, R. Ramaswamy, and M. Barma. Pairwise balance and invariant measures for generalized exclusion processes. *J. Phys. A: Math. Gen.*, 29:837, 1996.
37. A.M. Povolotsky and V.B. Priezzhev. Determinant solution for the totally asymmetric exclusion process with parallel update: II. ring geometry. *J. Stat. Mech.*, 2007:P08018.
38. M. Henkel and G.M. Schütz. Finite-lattice extrapolation algorithms. *J. Phys. A: Math. Gen.*, 21:2617–2633, 1988.
39. S. Prolhac. In preparation.
40. G.H. Hardy. *Divergent series*. Clarendon press, Oxford, 1949.
41. K. Matsumoto. On analytic continuation of various multiple zeta-functions. In *Number Theory for the Millennium II*, pages 417–440, 2002.